Chapter 2: Straight Line

2.7. On The Sign of the Expression *ax + by + c*

We demonstrated in Section 2.6 that the perpendicular distance p' of a point (x_1, y_1) from the line $ax + by + c = 0$ is

$$p' = -\left(\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right).$$

If p' is positive, or, $ax_1 + by_1 + c < 0$, the perpendicular from the origin $R(x_1, y_1)$ lies in the first or second quadrant of this co-ordinate system with $R(x_1, y_1)$ as origin. Let us try to visualize this result in Figures 2.14(a) and 2.14(b).

One may also say that if $ax_1 + by_1 + c < 0$, the point $R(x_1, y_1)$ lies 'below' the line $ax + by + c = 0$, see Figure 2.14(a).

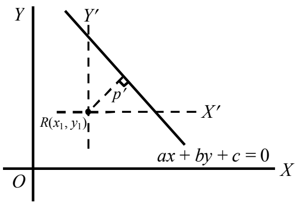


Figure 2.14(a)

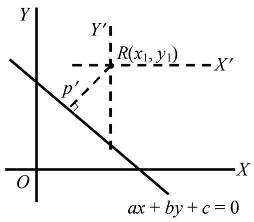


Figure 2.14(b)

Similarly, if p' is negative, or $ax_1 + by_1 + c > 0$, the perpendicular from the origin $R(x_1, y_1)$ on the line $ax + by + c = 0$ lies in the third or fourth quadrant of this co-ordinate system, Figure 2.14(b). Or, one may say that if $ax_1 + by_1 + c > 0$, the point $R(x_1, y_1)$ lies 'above' the line $ax + by + c = 0$.

It should be noted that these results hold only for the positive values of the co-efficient of y. On a general note, the straight line $ax + by + c = 0$ divides the co-ordinate plane into two parts such that $ax_1 + by_1 + c > 0$ for all points (x_1, y_1) in one region and $ax_1 + by_1 + c < 0$ for all the points (x_1, y_1) in the other region.

We may also generalize this analysis as follows:

Consider the relation $ax + by + c$ as a function in x and y such that

$f(x, y) = ax + by + c.$

We have established that $f(x, y) = 0$ for all the co-ordinates (x, y) that lie on the straight line $ax + by + c = 0.$

Further, any two points $M(x_1, y_1)$ and $N(x_2, y_2)$ lie on the same side of the straight line $ax + by + c = 0$ if $f(x_1, y_1) \cdot f(x_2, y_2) > 0$, Figure 2.14(c),

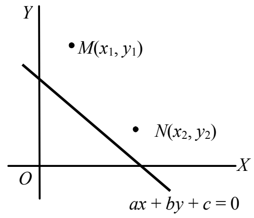


Figure 2.14(c)

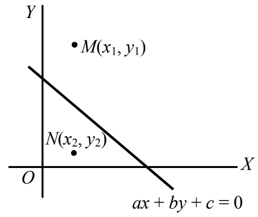


Figure 2.14(d)

that is, for all the points lying on the same side of the straight line, either $f(x, y) > 0$ or $f(x, y) < 0$. Nevertheless, the product of $f(x, y)$ for two points on the same side remains positive. Similarly, for any two points $M(x_1, y_1)$ and $N(x_2, y_2)$ lying on the opposite side of the straight line $ax + by + c = 0$, we have $f(x_1, y_1) \cdot f(x_2, y_2) < 0$, Figure 2.14(d).

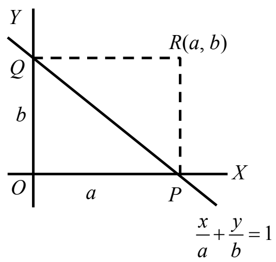
This idea finds its applications in many interesting problems. Study the example below to acquaint yourself better with the application.

Example 7. A straight line through a fixed point $(2, 3)$ intersects the co-ordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, find the locus of R.

This problem does not require any calculations, provided one starts with the variables that directly lead to the answer. Consider a straight line with intercepts a and b on the x-axis and y-axis respectively.

Consider the points P and Q where the straight line intersects the x-axis and the y-axis respectively. If we complete the square $OPRQ$, the co-ordinates of the vertex R are given by $R(a, b)$, where a and b are variables.

The equation of the line PQ in the intercept form is



$$\frac{x}{a} + \frac{y}{b} = 1.$$

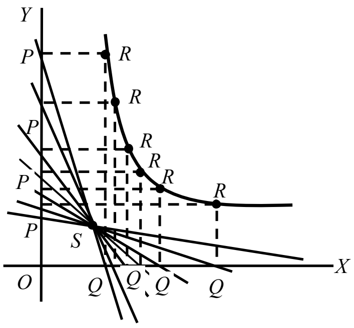
The line PQ always passes through the point $S(2, 3)$, therefore,

$$\frac{2}{a}+\frac{3}{b}=1$$

or $3a+2b=ab$.

On generalizing we get that the locus of $R(a,b)$ is the curve

$3x+2y=xy$.



The problem is almost a non-brainer, but the approach matters! It is advised to the student to revisit this problem after reading the chapter on hyperbola, Chapter 7. We will now endeavour to visualize the result.

If you look carefully, the locus of R represents a rectangular hyperbola with centre at $(2,3)$ given by the equation $(x-2)(y-3)=6$. The right arm of the curve is shown in the figure.

Example 2. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C, if the centre of C_1 lies at $(0,0)$ and the centre of C_2 lies at (a,b).

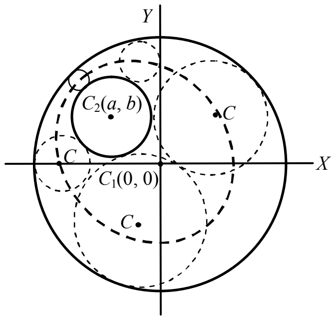
Let the co-rodinates of centre of the circle C be (h,k) and radius be equal to r. Further, we shall assume the radii of C_1 and C_2 to be equal to r_1 and r_2 respectively.

Since the circles C_1 and C touch internally, we have

$$\sqrt{h^2+k^2}=r_1-r. \qquad \ldots\text{(i)}$$

The circles C_2 and C touch externally, therefore,

$C_2C=r_2+r$



or $\sqrt{(h-a)^2+(k-b)^2}=r_2+r. \qquad \ldots\text{(ii)}$

On adding Eqs. (i) and (ii) we obtain

$$\sqrt{h^2+k^2}+\sqrt{(h-a)^2+(k-b)^2}=r_1+r_2$$

and, on generalizing, we obtain locus of centre of circle C,

$$\sqrt{x^2+y^2}+\sqrt{(x-a)^2+(y-b)^2}=r_1+r_2. \qquad \ldots\text{(iii)}$$

We may read Eq. (iii) as follows:

The point (x,y) traces a path such that the sum if its distances from two fixed points $(0,0)$ and (a, b) is always constant, equal to r_1+r_2. You will learn in Chapter-6 that this is the definition of an ellipse. Thus, the locus of centre $C(h,k)$ is an ellipse with foci at $(0,0)$ and (a, b), see the accompanying figure.