Chapter 6: Ellipse

6.7.1. Conjugate Diameters

We demonstrated in the previous section that all chords parallel to the line $y = mx$ are bisected by the line $y = m'x$, where

$$m' = -b^2 / a^2 m, \text{ or } mm' = -\frac{b^2}{a^2}. \qquad \ldots(1)$$

Take a look at the Eq. (1) again. The equation is symmetric with respect to m and m', and, therefore, demonstrates that all chords parallel to the line $y = m'x$ are bisected by the line $y = mx$. We can say that the locus of the mid-points of chords parallel to $y = m'x$ is $y = -\frac{b^2 x}{a^2 m'}$, that is, $y = mx$, since $-\frac{b^2}{a^2 m' = m}$.

Therefore, every line through the centre of an ellipse is a diameter; and if the slopes m and m' of any two diameters, $y = mx$ and $y = m'x$, are connected by the relation (1), each bisects all chords parallel to the other. Two such diameters are thus said to be conjugate. We shall now derive the equation and the co-ordinates of end points of a conjugate diameter.

Let $P(x', y')$ be any point on an ellipse with centre at the origin $O(0,0)$. Consider OP the diameter through P, and OQ the diameter conjugate to OP. The slope of OP is y'/x', and, therefore, the slope of OQ (conjugate to OP) is $-x'b^2 / y'a^2$. The equation of OQ is therefore,

$$y = -\frac{x'b^2}{y'a^2} x,$$

or $$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0. \qquad \ldots(1)$$

From this result it follows that OQ is parallel to the tangent at P, where the equation of tangent at P is given by $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$. Further, the diameter OP when extended through O meets the ellipse again in the point P', such that the co-ordinates of P' are $(-x', -y')$. Refer Figure 6.15.

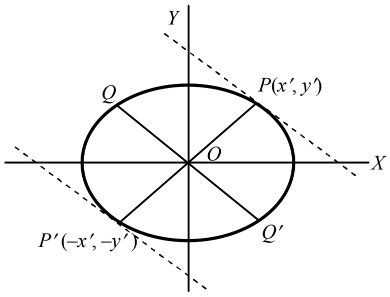


Figure 6.15

Extending the same argument, the tangent at P', given by $\frac{-xx'}{a^2} - \frac{yy'}{b^2} = 1$ is parallel to the conjugate diameter OQ.

It follows that the diameter of a system of parallel chords passes through the points of contacts of the tangents which are parallel to the chords. This can be seen by considering the tangents drawn at P and P'. Since tangents are chords which intersect the curve in two co-incident points, consider a system of chords drawn parallel to the tangents at P and P'. The diameter of this system of parallel chords is POP', which passes through the points of contacts of the tangents which are parallel to the chords, where one of the parallel chords is the conjugate diameter QOQ'. Refer figure 6.16(a).

Similarly, consider a system of parallel chords drawn parallel to the tangents at Q and Q'. The diameter of this system is QOQ', which passes through the points of contacts Q and Q' respectively, refer figure 6.16(b), where one of the parallel chords in the system is the conjugate diameter POP'.

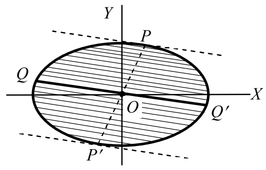


Figure 6.16(a)

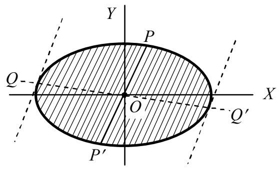


Figure 6.16(b)

We now proceed to find the coordinates of the points Q and Q', where the diameter conjugate to OP cuts the ellipse, in terms of the coordinates of $P(x', y')$.

The elimination of y/b from $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0$ (equation of QQ') gives

$$\frac{y'^2}{b^2}\left(1 - \frac{x^2}{a^2}\right) = \frac{x'^2}{a^2}\frac{x^2}{a^2},$$

or $$\frac{y'^2}{b^2} = \frac{x^2}{a^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right).$$

Since $P(x', y')$ lies on the ellipse, we have $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\equiv 1.$

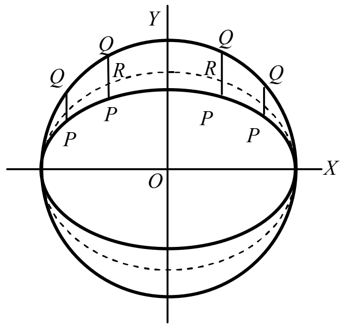
Thus, we get $\frac{x^2}{a^2}=\frac{y'^2}{b^2}$, or $\frac{x}{a}=\pm\frac{y'}{b}$; and this value of $\frac{x}{a}$ on substitution in the equation of QQ' gives $\frac{y}{b}=\mp\frac{x'}{a}$.

Therefore, the coordinates of Q and Q' are given by $\left(\pm y'\frac{a}{b}, \mp x'\frac{b}{a}\right)$; or we can say that Q is the point $\left(-y'\frac{a}{b}, x'\frac{b}{a}\right)$ and Q' is the point $\left(y'\frac{a}{b}, -x'\frac{b}{a}\right)$.

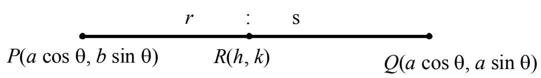
We will now discuss some of the properties of the conjugate diameters.

Example 7. Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $0<b<a$. Let the line passing through P and parallel to the y-axis meet on the circle $x^2+y^2=a^2$ at the point Q such that P and Q are on the same side of the x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR:RQ=r:s$ as P varies over the ellipse.

If you start by drawing a good diagram, you can guess the answer immediately. The locus of R is an ellipse with the major axis along the x-axis. We shall now proceed to validate this speculation analytically.



Consider a point $P(a\cos\theta, b\sin\theta)$ on the ellipse and correspondingly, point Q has the co-ordinates $(a\cos\theta, a\sin\theta)$. The point $R(h,k)$, divides the segment PQ in the ratio $r:s$.



Therefore, $h=\frac{a\cos\theta(r)+a\cos\theta(s)}{r+s}=a\cos\theta$

and $k=\frac{a\sin\theta(r)+a\sin\theta(s)}{r+s}=\left(\frac{ar+bs}{r+s}\right)\sin\theta.$

Using $\cos^2\theta+\sin^2\theta=1$, we obtain

$$\left(\frac{h}{a}\right)^2+\left(\frac{r+s}{ar+bs}\right)^2\cdot k^2=1.$$

On generalizing, we get

$$\left(\frac{x}{a}\right)^2+\left(\frac{r+s}{ar+bs}\right)^2\cdot y^2=1$$

which represents an ellipse with the semi major axis equal to a units, and semi-minor axis equal to $\left(\frac{ar+bs}{r+s}\right)$ units.