Chapter 3: Circle

3.5. Equation of a Chord in terms of its Middle Point

If $M(x_1, y_1)$ be the mid point of a chord of the circle $x^2 + y^2 = a^2$, then the equation of the chord is given by $xx_1 + yy_1 = x_1^2 + y_1^2$.

And in general, if $M(x_1, y_1)$ be the mid point of a chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

We shall prove this result analytically by two methods.

Method 1: Consider a chord QR of the circle with centre $C(-g, -f)$. The co-ordinates of the mid point of the chord are given by $M(x_1, y_1)$, see Figure 3.15.

The chord QR is perpendicular to the line joining the centre C and the mid-point M. (How? Can you see two congruent triangles CMQ and CMR?

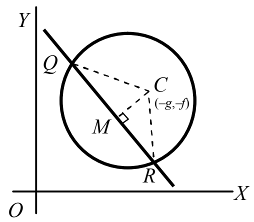


Figure 3.15

If the triangles are congruent, $\angle CMQ = \angle CMR = 90^\circ$.)

Slope of $CM = \dfrac{y_1 + f}{x_1 + g}$.

Therefore, slope of $QR = -\left(\dfrac{x_1 + g}{y_1 + f}\right)$.

Thus, the equation of the chord QR is given by

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

or $xx_1 + yy_1 + fy + gx = x_1^2 + y_1^2 + gx_1 + fy_1$.

On adding $gx_1 + fy_1 + c$ to both the sides, we get

$$xx_1 + yy_1 + g(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

or $T = S_1$,

which is the required equation.

Method 2: Let the equation of the chord QR be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

where θ is the inclination of QR with respect to the positive direction of x-axis and r is the algebraic distance of any point (x, y) on the chord from the mid-point $M(x_1, y_1)$.

Substitute

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta$$

into the equation of the circle, to get

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 = a^2$$

or $r^2 + 2r(x_1\cos\theta + y_1\sin\theta) + x_1^2 + y_1^2 - a^2 = 0$.

The equation gives two solutions for r, corresponding to the two points of intersection of the chord with the circle. Now since $M(x_1, y_1)$ is the mid point of the chord, the values of r furnished by this equation must be equal in magnitude and opposite in sign. (Why? Because Q and R are located symmetrically about M on the given chord.) Or, one may say that the sum of the roots of the equation is equal to zero. That is, $x_1\cos\theta + y_1\sin\theta = 0$

or $\tan\theta = \dfrac{-x_1}{y_1}$.

Therefore, the equation of the chord is given by

$$y - y_1 = \frac{-x_1}{y_1}(x - x_1)$$

or $xx_1 + yy_1 = x_1^2 + y_1^2$.

Example 7. A circle touches the x-axis and also touches the circle with centre at $(0,3)$ and radius 2. Find the locus of centre of the circle.

Consider a circle with centre $C_1(0,3)$ and radius $r_1 = 2$ units. The equation of this circle is given by

$$x^2 + (y - 3)^2 = 4.$$

Now consider a circle which touches the x-axis and the circle we described above. Let the co-ordinates of the centre of this circle be given by $C_2(h, k)$ and radius r_2 units. The circle touches the x-axis, $k = r_2$.

Since the circles touch externally, we can write

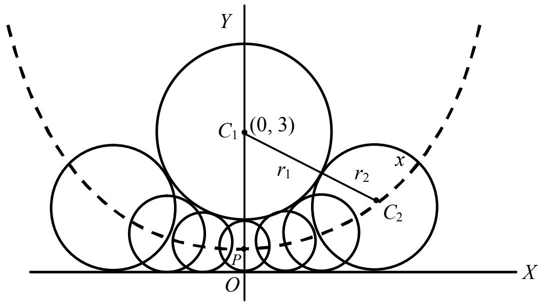
$C_1C_2 = r_1 + r_2$

or $\sqrt{h^2 + (k - 3)^2} = 2 + k$

or $h^2 = 10\,k - 5 = 10\left(k - \dfrac{1}{2}\right)$.

On generalizing we get $x^2 = 10\left(y - \dfrac{1}{2}\right)$.

We shall see in Chapter 5 that the locus of $C_2(h, k)$ is a parabola with vertex at the point $P\left(0, \dfrac{1}{2}\right)$. You may visualize the result in the accompanying figure; the locus of the centre is shown by dotted line.



Y
C_1 (0, 3)
r_1
r_2
x
C_2
P
O
X