Sample Booklet

Chapter - 4

Motion in Two Dimensions

Rectilinear motion is only a special case - the simplest one indeed. But the bodies do not always move in straight lines. A roller coaster employs an elevated railroad track with tight turns, steep slopes and sometimes inversions.

4.1. Motion in a Plane

Before going into the specifics of motion in a plane we reproduce two paragraphs from Chapter - 3 to begin with.

Let us consider an example from everyday life. Make a mark on the rim of your bicycle and focus on the mark when you ride it. What is the trajectory of the mark as seen by you? Is it the rim circumference itself? What is the trajectory of the mark relative to a person standing on the ground? Figure 3.2 shows the trajectory of a point (P) on the rim of a rolling wheel relative to the ground.

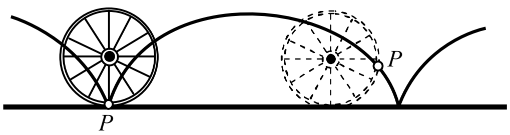


Fig. 3.2

To describe the motion of a body, we must know how its various points move. Many a time we are interested only in the change of the position of the body as a whole. Consequently, in some cases the description of the motion of a body is reduced to the description of the motion of a point.

Various types of motion of a point differ first of all in the shape of the paths. If the path is a straight line, the motion of the point is referred to as *rectilinear* (*motion in a straight line or motion in one dimension*). If the path is a curve, the motion is said to be *curvilinear* (motion in two or three dimensions). For instance, the centre of the wheel rolling on a horizontal road in Fig. 3.2 moves in a straight line, while point P is in a curvilinear motion.

Point P in the above example moves in a plane, so does a piece of stone thrown obliquely in air, the swinging bob of a simple pendulum and a girl in a merry-go-round. Can you give a few more examples? In this chapter we shall study the motion of bodies in a two-dimensional plane.

4.2. Displacement, Velocity and Acceleration

The quantities like displacement, velocity, and acceleration were introduced in Chapter - 3 for one-dimensional motion. We must generalize them to two dimensions emphasizing their vector nature.

Let us consider a simple example. As depicted in Fig. 4.1, a girl strolling on a ground moves from point A to point B along a path (Path *1*) of length 60 m. After some time she finds herself at point A again. She goes to point B for the second time, now along a different path (Path *2*) of length 40 m. After a while the girl again walks from point A to point B, covering a 90-m distance along Path *3*.

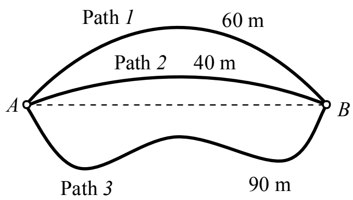


Fig. 4.1

In the three laps of motion described above the girl moves from point A to point B - along three different paths covering three different distances.

What is common to these three laps of motion of the girl? The staring point A, from where she starts moving and point B, where she halts. We can say that the initial position of her motion is point A and the final position is B. Her position changes from point A to point B, in each lap.
It turns out that this *change in position* is an extremely useful quantity in describing the motion of bodies. We call this quantity *displacement*. Thus displacement is *change in position* of a moving body.

We list here a few incorrect definitions of displacement we have been hearing from students over the years.
*Displacement is *the shortest distance between two points.*
*Displacement is *the shortest distance between the initial and final positions of a moving body.*
*Displacement is *the straight line connecting the initial and final positions of a moving body.*
These statements do not qualify to be definition of a physical quantity, the clauses in them do not signify something essentially meaningful. These clauses assume specific meanings in certain situations. Students are required to give these statements a thought and figure out why they cannot be definition of displacement.
But as we understand, science is not about coining words and definitions. It is about experiments, measurements, formulations, interpretations, applications and predicting possible outcomes. Here naturally arise some important questions: How to represent displacement? How to measure it? How to use it in describing the motion of a body? How to relate it to other kinematical quantities?
In the example of the strolling girl described above (Fig. 4.1), her position changes from point A to point B (see the dotted line). The change in the position can be represented by an arrow placed between points A and B, the tail of the arrow at point A, representing the initial position, and its head at point B, the final position.
The length of the arrow tells *by how much* did the position change, that is, the value (more formally called the magnitude) of the displacement and the tail-head orientation indicates the *direction in which* this change occurred. Displacement has a direction also - it is in the direction in which the head of the arrow points.
Hence, the displacement of a moving body must tell you two things: by how much did the position change and in which direction did the change occur. Only the value of displacement or only its direction is not enough.
Before we proceed further, we must understand another characteristics of this quantity. Suppose the girl strolling on the ground goes from point A to point B and then from point B to point C, as depicted in Fig. 4.2.

What is the total distance the girl traveled? It is the sum of the lengths of the paths *1* and *2*: total distance is $d = 60\text{ m} + 50\text{ m} = 110\text{ m}$.

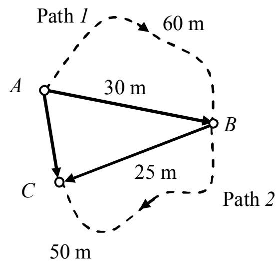


Fig. 4.2

What is the net change in the position of the girl? How to represent it? It can be represented by an arrow drawn from her initial position A to the final position C. Is the length of this arrow equal to the sum of the lengths of the arrows AB and BC, $30\text{ m} + 25\text{ m} = 55\text{ m}$? Most unlikely. It follows that displacements cannot be added numerically or algebraically. They belong to some other *class* with its own set of rules for manipulation.
Let us represent the three displacements, from A to B, from B to C and from A to C as $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively. Also, as an example, assume that the length of the line segment AC is 15 m. Then,

$$\underset{30\text{ m}}{\overrightarrow{AB}} + \underset{25\text{ m}}{\overrightarrow{BC}} = \underset{15\text{ m}}{\overrightarrow{AC}}.$$

Here emerges an entirely different way of addition: a displacement in one direction when added to a displacement in another direction gives a displacement in a third direction whose magnitude may be different from the sum of the magnitudes of the displacements added. Look at the arrangement of the arrows along the sides of the ΔABC in the figure carefully. This law of addition of physical quantities, called the triangle law or parallelogram law (as explained in Chapter 2), which is followed by displacement and many other physical quantities is the *vector law of addition*; and the quantities that follow this law are christened *vectors*.
We shall now learn how to represent and work with displacements in a systematic way. In our example (Fig. 4.1), the position of the girl changed from point A to B. It makes sense to represent her position, and also the change in position in a more specific, mathematical way. A convenient way to do it is to draw a position vector from a reference point.
In a two dimensional plane the position vector $\vec{r}$ of a point whose coordinates are (x, y) is represented as $\vec{r} = x\hat{i} + y\hat{j}$, as shown in Fig. 4.3.
The Cartesian coordinates x and y are called the scalar components of the position vector $\vec{r}$, and $\hat{i}$ and

$\hat{j}$ are unit vectors in the direction of x- and y- axes respectively.

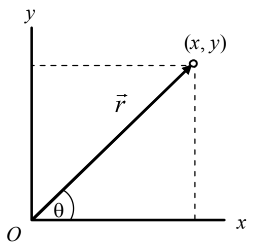


Fig. 4.3

We can also represent the position of a point by r (length of the line drawn from origin to the point) and θ (angle the line makes, say, with x- axis in anticlockwise sense). The two descriptions of the position vector are equivalent in the sense we can pass back and forth between them. If we are given (x, y), we can find r and θ from

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x},$$

and in case (r, θ) are known, we can obtain x and y from

$$x = r\cos\theta, \; y = r\sin\theta.$$

If a point moves from position A whose position vector is $\vec{r}_1$ to position B with position vector $\vec{r}_2$, as shown in Fig. 4.4, the displacement, that is, the change in position, is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \Delta x\,\hat{i} + \Delta y\,\hat{j}.$$

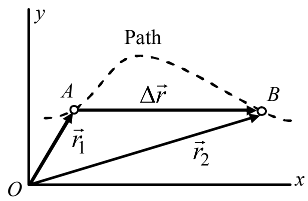


Fig. 4.4

The following should help you to draw the direction of $\Delta\vec{r}$ correctly: $\Delta\vec{r}$ is the vector that must be added to the initial position vector $\vec{r}_1$ to give the final position vector $\vec{r}_2$, that is, $\vec{r}_1 + \Delta\vec{r} = \vec{r}_2$.

Now we shall correlate the displacement of a motion to the corresponding time interval which leads to the concept of velocity. As in Chapter - 3, the average velocity is defined as the ratio of the displacement to the time interval over which the displacement occurred.

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}. \quad \text{...(4.1)}$$

It can be readily seen that the magnitude of the average velocity is obtained by dividing the magnitude of displacement by the time interval. What about its direction? Which direction should be ascribed to average velocity? The direction of average velocity $\vec{v}_{av}$ is that of $\Delta\vec{r}$, which is directed along a chord across the path shown in the above figure.

The concept of average velocity has limited application as it lacks in details of motion. Many times we are interested in velocity of a particle when it is at a certain position or at an instant of time. This necessitates developing the concept of instantaneous velocity. While calculating average velocity the shorter the intervals we choose, the smaller is the difference between the corresponding small segment of the path and its chord. For a sufficiently small path length, the chord will be practically indistinguishable from the tangent drawn at any point of this segment of the path. The direction of instantaneous velocity is the direction of the tangent at the point of the path where the moving point is at a given instant of time.

The instantaneous velocity is

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

which in the notation of calculus is written as

$$\vec{v} = \frac{d\vec{r}}{dt}. \quad \text{...(4.2)}$$

We can express instantaneous velocity in terms of its components as

$$\vec{v} = v_x\hat{i} + v_y\hat{j},$$

where $v_x = \frac{dx}{dt}$, and $v_y = \frac{dy}{dt}$. The direction of $\vec{v}$ is along the tangent to the path, see Fig. 4.5.

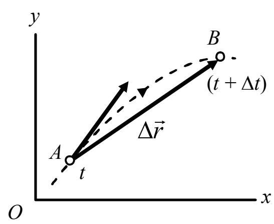


Fig. 4.5

An important point to be noted is that the instantaneous velocity $\vec{v}$ is directed along the tangent to the path, but its magnitude is *not* the slope of that line. Why? Notice that the diagram is not a position-time graph. Consequently, the magnitude of $\vec{v}$, the instantaneous speed, is not given by the slope of the tangent.

For a point moving in one dimension, a change in velocity can be effected only be changing its value; it can be increased or decreased. When the velocity of the point reverses its sign, the direction of motion of the point is reversed. In case of motion in two dimensions, as well as in three-dimensional space, there arise many possibilities.

For a point in planar motion, its velocity may change only in magnitude or only in direction or in most of the cases, in both magnitude and direction.

Thus, in a curvilinear motion the velocity continuously changes, which means that the point moves with an *acceleration.* To determine this acceleration (its magnitude and direction), we have to find the change of velocity *as a vector,* i.e., we have to determine the change of the magnitude of velocity and the direction of change in velocity.

Suppose that a point has a velocity $\vec{v}_1$ at an instant of time t_1 and $\vec{v}_2$ after a time interval Δt, that is to say at instant t_2. The change in velocity is obtained by subtracting vector $\vec{v}_1$ from $\vec{v}_2$. The *average acceleration* is the ratio of the velocity change to the interval of time Δt over which this change occurs.

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta \vec{v}}{\Delta t}. \qquad \ldots(4.3)$$

The direction of average acceleration $\vec{a}_{av}$ coincides with the direction of the change in velocity vector $\Delta \vec{v}$. Does the direction of $\Delta \vec{v}$ coincide with the direction of $\vec{v}_2$? Can it? What if the point starts from rest?

The direction of acceleration of a point moving in a plane does not coincide, in general, with the direction of velocity. In order to find the direction of acceleration, we compare the directions of the velocities at two close points on the path. Since the velocities are directed along the tangents to the paths, the direction of acceleration can be determined from the shape of the path. Since the change $\vec{v}_2 - \vec{v}_1$ of the velocities at two close points is always directed towards the bending of the path, it means that acceleration is always directed inside of the curved path, see Fig. 4.6.

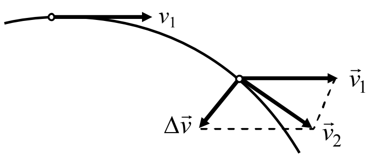


Fig. 4.6

Since average accelerations of a motion computed over different time intervals are not necessarily same, it warrants to develop the concept of instantaneous acceleration: acceleration at a certain point of the path or acceleration at a certain instant of time.

By choosing a sufficiently small Δt, we arrive at the concept of *instantaneous acceleration.* The instantaneous acceleration is the rate of change of velocity relative to time:

$$\vec{a} = \frac{d\vec{v}}{dt} = a_x \hat{i} + a_y \hat{j} \qquad \ldots(4.4)$$

where $a_x = \frac{dv_x}{dt}$, and $a_y = \frac{dv_y}{dt}$.

In passing it is worth noting that it is far more convenient to calculate the two components a_x and a_y with the help of calculus or otherwise than calculating vector $\vec{a}$.

Can we determine acceleration $\vec{a}$ directly from the path of the point? No. We need to know how each component of the velocity varies as a function of space and time. Figure 4.7 shows possible directions for the acceleration of a point that travels along a curved path with varying speed.

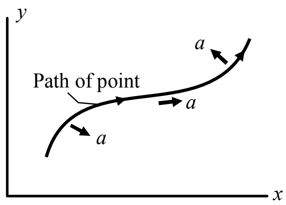


Fig. 4.7

▶ **Example 1.** The position vector $\vec{r}$ of a particle is given by the following equation $\vec{r}(t) = \alpha t^3 \hat{i} + \beta t^2 \hat{j}$, where $\alpha = \frac{10}{3}$ ms^{-3} and $\beta = 5$ ms^{-2}. At $t = 1$ s, what are the velocity and acceleration of the particle?

The derivative of the position vector function gives the velocity vector and the derivative of the velocity vector function gives the acceleration, (Eqs. (4.2) and (4.4)).

$$\vec{v}(t) = 3\alpha t^2 \hat{i} + 2\beta t\, \hat{j}$$

$$\vec{a}(t) = 6\alpha t\, \hat{i} + 2\beta\, \hat{j}.$$

Substituting the given values of α and β, and $t = 1$ s into the expressions for the position vector, velocity and acceleration, we obtain

$$\vec{r} = \frac{10}{3}\hat{i} + 5\hat{j} \text{ m}$$

$$\vec{v} = 10\hat{i} + 10\hat{j} \text{ m/s}$$

$$\vec{a} = 20\hat{i} + 10\hat{j} \text{ m/s}^2.$$

❓ At $t = 1$ s, what is the angle between $\vec{r}$ and $\vec{v}$? Between $\vec{r}$ and $\vec{a}$? Between $\vec{v}$ and $\vec{a}$?

We can use the dot product of two vectors (Eq. 2.4, Chapter 2) to compute these angles. But in this problem you know the *x*- and *y*- components of the position velocity and acceleration vectors. You can find the required angles straightaway. All you need to do is to draw these vectors on *x-y* plane (Fig. 4.8), and use elementary trigonometry.

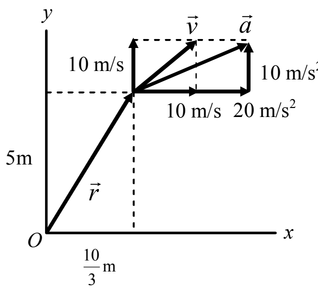


Fig. 4.8

4.3. Motion in a Plane with Constant Acceleration

Let us consider the case of two dimensional motion of a point in which the acceleration does not vary either in magnitude or in direction. In that case the components of acceleration $\vec{a}$ in *any* Cartesian coordinate system will not vary, that is, a_x = constant and a_y = constant. *Under this condition the motion of the point can be described as the sum of two component motions occurring simultaneously with constant acceleration along each of the two axes*. This simplifies the analysis amazingly. The point will move, in general, along a curved path in the plane. Will the point move on a curved path even if one component of the acceleration, say a_x, is zero? Can one component of the velocity, say v_x, have a constant, non-zero value? What about the motion of a cricket ball which follows a curved path in a vertical plane when the effect of air resistance is neglected?

We can obtain the general equations for two dimensional motions with constant acceleration simply by setting

a_x = constant and a_y = constant.

The equations for constant acceleration (Eqs. 3.7, through 3.10, Chapter 3) then apply separately and independently to both the x- and y- components of the displacement vector $\vec{S}$, the velocity vector $\vec{v}$, and the acceleration vector $\vec{a}$. From this idea we generate two sets of equations given below.

(i) Equations for motion in the direction of x- axis:

$$v_x = u_x + a_x t \qquad \ldots(4.5a)$$

$$S_x = u_x t + \frac{1}{2} a_x t^2 \qquad \ldots(4.5b)$$

$$x = x_0 + u_x t + \frac{1}{2} a_x t^2 \qquad \ldots(4.5c)$$

$$v_x^2 = u_x^2 + 2a_x S_x. \qquad \ldots(4.5d)$$

(ii) Equations for motion in the direction of y- axis:

$$v_y = u_y + a_y t \qquad \ldots(4.6a)$$

$$S_y = u_y t + \frac{1}{2} a_y t^2 \qquad \ldots(4.6b)$$

$$y = y_0 + u_y t + \frac{1}{2} a_y t^2 \qquad \ldots(4.6c)$$

$$v_y^2 = u_y^2 + 2a_y S_y. \qquad \ldots(4.6d)$$

Now the most important question is how the two sets of equations are related? The answer to this question lies in that the time parameter t is the same for each, since t represents the time at which the point occupied a position defined by the co-ordinates x and y.

The equations of motion in a plane may also be expressed in vector form. For example, velocity of the point at a variable time t can be written in terms of its components as

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$
$$= (u_x + a_x t)\hat{i} + (u_y + a_y t)\hat{j}$$
$$= (u_x\hat{i} + u_y\hat{j}) + (a_x\hat{i} + a_y\hat{j})t.$$

The first quantity in parentheses is the initial velocity vector $\vec{u}$ and the second is the (constant) acceleration vector $\vec{a}$ multiplied by t. Thus, the vector relation

$$\vec{v} = \vec{u} + \vec{a}t \qquad \ldots(4.7)$$

is equivalent to the two scalar relations. This relation shows that the velocity $\vec{v}$ at time t is the sum of the initial velocity $\vec{u}$ and the (vector) change in velocity, $\vec{a}t$, during the time interval t under the constant acceleration $\vec{a}$. Similarly, the vector equation for the displacement is

$$\vec{S} = \vec{u}t + \frac{1}{2}\vec{a}t^2. \qquad \ldots(4.8)$$

One can easily interpret this equation by ascribing meaning to the two terms that appear on its right hand side.

We shall now apply the concepts discussed above to some common motions, viz. projectile motion, circular motion, relative motions in two dimensions, and so on.

4.4. Projectile Motion

The fact that the laws of natural sciences are non-intuitive in nature is often reflected in our everyday life. For example, for the motion of a ball thrown in air, many of us believe that the force used to throw the ball up somehow stays with it. The 'force of the hand' is supposed to be gradually overcome by the force of gravity, which ultimately causes the ball to fall. It is not surprising that as late as the sixteenth century it was believed that when a shell was fired, it was given an 'impressed force' that produced 'violent' motion in a straight line, thereafter followed a region of mixed motion ('violent' plus 'natural' motion vertically down) because of air resistance, and finally, the 'natural' motion vertically down prevailed, see Fig. 4.9. Initially, Galileo also believed that the motion of a projectile was governed by an 'impressed force' that gradually diminished. It was only after he had developed his principle of inertia that he could tackle the problem of projectile motion properly.

Galileo had arrived at the crucial insight that a projectile near the surface of the earth has two *independent motions:* A horizontal motion at constant speed and a vertical motion subject to the acceleration due to gravity.

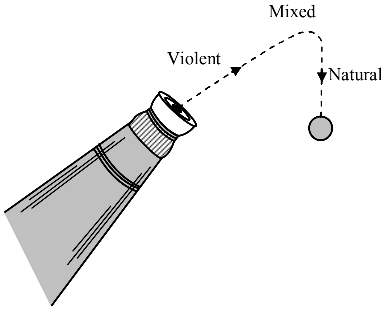


Fig. 4.9

This very common form of motion is surprisingly simple to analyze if two facts are taken into consideration: (i) The acceleration due to gravity g is constant over the range of motion and is directed downward. (ii) The effect of air resistance is negligible, which makes us to assume that there is no acceleration in the horizontal direction.

If we choose our reference frame such that the y- direction is vertical and positive upward, then $a_y = -g$ (as in one-dimensional free fall) and $a_x = 0$ (since air resistance is neglected). Suppose that at $t = 0$ the projectile leaves the origin (0, 0) with a velocity v_0, as in Fig. 4.10. If the velocity vector makes an angle θ (*angle of projection*) with the horizontal, the initial x- and y- components of velocity are given by

$$v_{0x} = v_0 \cos\theta \text{ and } v_{0y} = v_0 \sin\theta.$$

Substituting these expressions into Eqs. 4.5 and 4.6 with $a_x = 0$ and $a_y = -g$ gives the velocity components and coordinates for the projectile at any time t.

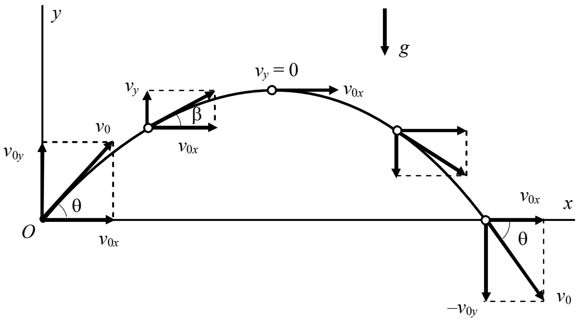


Fig. 4.10

$$v_x = v_0 \cos\theta \qquad \ldots(4.9)$$

$$v_y = v_0 \sin\theta - gt \qquad \ldots(4.10)$$

$$x = (v_0 \cos\theta)t \qquad \ldots(4.11)$$

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2. \qquad \ldots(4.12)$$

By a look at these equations certain observations can be made immediately. We see that v_x remains constant in time and is equal to the initial x- component of velocity, since there is no horizontal component of acceleration. Also, for the motion in y- direction, we note that expressions for v_y and y are identical to that for v_y and y for a particle thrown vertically upward and moving with constant acceleration g, as discussed in Section 3.5.2, Chapter 3.

If we solve for t in Eq. 4.11 and substitute the expression for t into Eq. 4.12, we find the equation of the trajectory:

$$y = (\tan\theta)x - \left(\frac{g}{2v_0^2 \cos^2\theta}\right)x^2, \qquad \ldots(4.13)$$

which is valid for the angle θ in the range $0 < \theta < \frac{\pi}{2}$.

Why? This equation is of the form $y = ax - bx^2$, which is the equation of a parabola that passes through the origin. You will learn about this function when you study quadratic equations in algebra or parabola in coordinate geometry.

It is extremely useful to investigate the motion of the projectile in a little detail and compute a few variables, like its velocity $\vec{v}$ as a function of time, the angle its velocity vector makes with the horizontal, its speed, its position vector and distance from the origin, the angle between its velocity vector and acceleration vector at certain instant t, and so on.

We can obtain the velocity $\vec{v}$ as a function of time for the projectile by noting that Eqs. 4.9 and 4.10 give the x- and y- components of velocity at any instant:

$$\vec{v} = (v_0 \cos\theta)\hat{i} + (v_0 \sin\theta - gt)\hat{j}.$$

Also, since the velocity vector is tangent to the path at any instant, as shown in Fig. 4.10, the angle β that $\vec{v}$ makes with the horizontal can be obtained from v_x and v_y through the expression

$$\tan\beta = \frac{v_y}{v_x}.$$

Further, by definition, the instantaneous speed v is equal to the magnitude of the instantaneous velocity $\vec{v}$, therefore,

$$v = \sqrt{v_x^2 + v_y^2}.$$

The expression for the position vector as a function of time t for the projectile follows directly from Eq. 4.8, with $\vec{a} = \vec{g}$ we have

$$\vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{g}t^2.$$

This expression is plotted in Fig. 4.11. It is interesting to note that the motion can be considered as the superposition of the term $\vec{v}_0 t$, which is the displacement if no acceleration were present, and the term $\frac{1}{2}\vec{g}t^2$, which arises from the acceleration due to gravity. In other words, if there were no gravitational acceleration the projectile would continue to move along a straight path in the direction of $\vec{v}_0$.

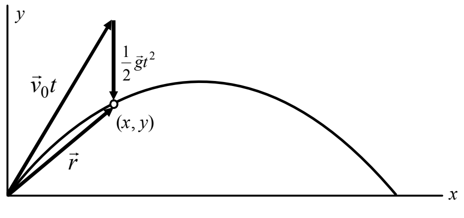


Fig. 4.11

We conclude that projectile motion is the superposition of two motions: (1) the motion of a freely falling body in the vertical direction with constant acceleration and (2) motion in the horizontal direction with constant velocity.

▶ **Example 2.** A hunter wishes to shoot a koala hanging from a branch. The hunter aims right at the koala, not realizing that the bullet will follow a parabolic path and thus will fall below the koala. The koala, however, seeing the firing, lets go of the branch and drops out of the tree, expecting to avoid a hit. Show that the koala will be hit regardless of the initial velocity of the bullet so long as it is large enough to travel the horizontal distance to the tree before hitting the ground.

Let the horizontal distance to the tree be x and the original height of the koala be H, as shown in Fig. 4.12. Then the gun is aimed at an angle given by tan θ = H/x. If there were no gravity, the bullet would reach the height H (with vertical velocity v_{0y}) in the time t taken for it to travel the horizontal distance x, that is,

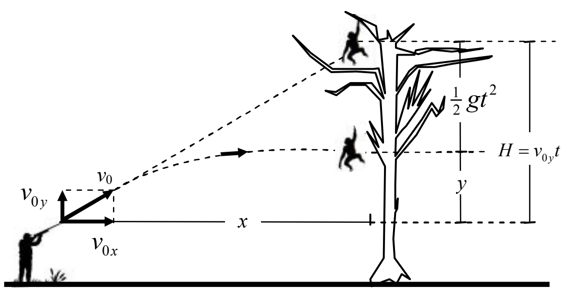


Fig. 4.12

$$v_{0y}t = H \text{ in time } t = \frac{x}{v_{0x}}.$$

However, because of gravity, the bullet has an acceleration vertically down. In time $t = x / v_{0x}$, the bullet reaches a height given by

$$y = v_{0y}t - \frac{1}{2}gt^2 = H - \frac{1}{2}gt^2.$$

This is lower than H by $\frac{1}{2}gt^2$, which is just the amount the koala falls in this time. For large v_0 the koala is hit very near its original height, and for small v_0 it is hit just before it reaches the ground. ◀

4.4.1. The Maximum Height and Horizontal Range of a Projectile

There are two points that are of special interest: the highest point of the trajectory with Cartesian coordinates (R/2, H) and the point with coordinates (R, 0) where the projectile comes in level with the launch point, see Fig. 4.13. The distance H is called the *maximum height* of the projectile and R its *horizontal range.*

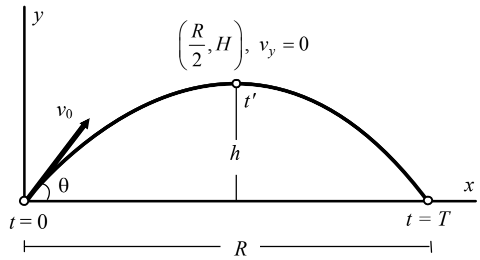


Fig. 4.13

We can determine the maximum height H reached by the projectile by noting that $v_y = 0$ at the peak. Therefore, Eq. 4.10 can be used to determine the time t' it takes to reach the peak:

$$t' = \frac{v_0 \sin\theta}{g}.$$

Substituting this expression for t' into Eq. 4.12 gives the maximum height H in terms of v_0 and θ:

$$H = v_0 \sin\theta\left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2$$

or $$H = \frac{v_0^2 \sin^2\theta}{2g}. \qquad \text{...(4.14)}$$

Alternatively, the maximum height H can be obtained from Eq. 4.6d directly,

$$0^2 = (v_0 \sin\theta)^2 - 2gH,$$

whence $H = \dfrac{v_0^2 \sin^2\theta}{2g}$.

The *range* R is the horizontal distance traveled in twice the time the particle takes to reach the peak, that is, in time $2t'$. Since the acceleration in the vertical direction is constant, the time of decent of the projectile must equal the time of ascent, therefore, the total *time of flight* is equal to

$$T = 2t' = \frac{2v_0 \sin\theta}{g}. \qquad \ldots(4.15)$$

This can also be seen that by setting $y = 0$ in Eq. 4.12 and solving the quadratic equation for t, one solution is $t = 0$, and the other solution is $t = \frac{2v_0 \sin\theta}{g} = T$. Using Eq. 4.11 and noting that $x = R$ at $t = T$, we find that

$$R = v_0 \cos\theta\left(\frac{2v_0 \sin\theta}{g}\right) = \frac{2v_0^2 \sin\theta\cos\theta}{g}$$

which is

$$R = \frac{v_0^2 \sin 2\theta}{g}. \qquad \ldots(4.16)$$

It is important to note that Eq. 4.16 is valid only when the projectile returns to the initial vertical level, that is, $\Delta y = 0$.

Keep in mind that Eqs. 4.14 and 4.16 are useful only for calculating H and R if v_0 and θ are known and only for a symmetric path, as shown in Fig. 4.10. The general expressions given by Eqs. 4.9 through 4.12 are the *most important* results, since they give the coordinates and velocity components of the projectile at *any* time t.

You should also note that the maximum value of R from Eq. 4.16 is $R_{\max} = v_0^2 / g$. This result follows from the fact that the maximum value of $\sin 2\theta$ is unity, which occurs when $2\theta = 90°$. Therefore, we see that the range R is a maximum when the angle of projection $\theta = 45°$, if resistance of air is neglected.

Figure 4.14 illustrates various trajectories for a projectile of a given initial speed.

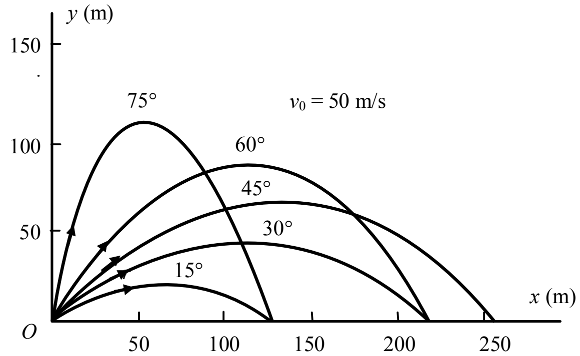


Fig. 4.14

As you can see, the range is maximum for $\theta = 45°$. In addition, for any θ other than 45°, a point with coordinates (R, 0) can be reached with *two* complementary values of θ, such as 75° and 15°. Of course, the maximum height and time of flight will be different for these two values of θ. Figure 4.14 illustrates, by way of example, that for two complementary angles, 75° and 15°, and 60° and 30°, the range of projectile is same. We used $g = 9.8\,\text{m/s}^2$ to generate the trajectories of the projectile launched at different angles.

To emphasize this fact let us consider another example. If $v_0 = 20$ m/s and $R = 30$ m, then

$$\sin 2\theta = \frac{Rg}{v_0^2} = \frac{(30\text{ m})\times(9.8\text{ m/s}^2)}{(20\text{ m/s})^2} = 0.735\,.$$

Thus $\theta = 23.7°$ or $66.3°$. Notice that $\theta = 45° \pm \alpha$, where $\alpha = 21.3°$, see Fig. 4.15.

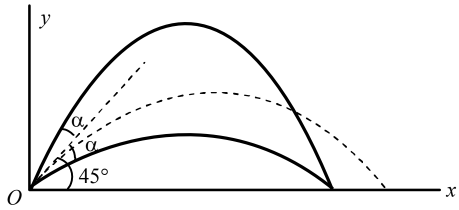


Fig. 4.15

❓ Find the maximum height reached and horizontal range if a projectile is launched at $50\,\text{m/s}$ at an angle (i) 15°, (ii) 30°, (iii) 45°, (iv) 60° and (v) 75° with the horizontal. Take $g = 9.8\,\text{m/s}^2$.

An important practical aspect of the motion of the projectiles must be understood. Computations based on the above equations are accurate only when air resistance can be neglected and the acceleration due to gravity is constant both in magnitude and in direction.

The predictions of these equations are valid only when the projectile is launched at a speed much less than its terminal speed (refer to Chapter 16). They do not really apply even to such commonplace projectiles as cricket and golf balls, let alone arrows, bullets, or ballistic missiles. They may be applied to slower projectiles, such as a shot-put. Nonetheless, in other cases they do provide a good first approximation to a complete, and usually far more complex solution.

The student should not feel compelled to memorize the expressions for time of flight, maximum height, and range or the equation of trajectory. These expressions are useful only in answering straightforward questions. The important thing is to follow the line of reasoning and steps used to obtain these results. The wisest way to handle problems on projectile motion is NOT TO USE THESE FORMULAE, EVER. You are not supposed to even remember them. You can anyway derive any of them in 8 to 10 seconds if need be.

It is far more fruitful to solve problems on projectile from the basics: draw a diagram, resolve the initial velocity into horizontal and vertical components, and then analyze the two components of motion separately and independently, using the familiar equations $v = u + at$, $S = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2aS$ for motion with constant acceleration. This makes the solution unimaginably simple.

▶ **Example 3.** A projectile is launched at moment $t = 0$ from point O on the ground with a velocity 20 m/s at an angle of 53° with the horizontal. Point O is origin of a coordinate system whose x- axis is horizontal and y- axis is directed vertically upward, see Fig. 4.16. Take the unit vectors along the x- and y- axes as $\hat{i}$ and $\hat{j}$ respectively. Take the acceleration due to gravity as 10 m/s² downward, that is, opposite to positive y- axis. Take sin 53° = 0.8, cos 53° = 0.6.

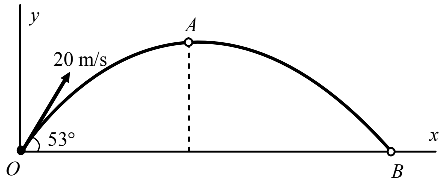


Fig. 4.16

(a) What is the total time of flight of the projectile?
You can calculate the time of flight directly from Formula 4.15 as

$$T = \frac{2v_0 \sin\theta}{g} = \frac{2\times 20 \text{ m/s} \times \sin 53^\circ}{10 \text{ m/s}^2} = 3.2 \text{ s}.$$

But as instructed above, you should follow the line of reasoning that was employed in deriving this and other formulas of projectile motion to solve even simple straightforward problems. You will understand the benefits of this habit when, after solving a variety of problems, you attain maturity and realize how beautifully this approach works for most complex problems.
The x- and y- components of the initial velocity of the projectile are $(20 \text{ m/s})\cos 53^\circ = 12$ m/s and $(20 \text{ m/s}) \sin 53^\circ = 16$ m/s, see Fig. 4.17.
At the apex A of the trajectory, the velocity has only horizontal component, while v_y vanishes. In order to find the instant t_A at which the projectile reaches point A, we substitute t_A for t into Formula 4.10 and equate the obtained result to zero.

$$(16 \text{ m/s}) - (10 \text{ m/s}^2)t_A = 0,$$

which gives $t_A = 1.6$ s.
Since the point from where the projectile was launched and the point on the ground where it lands are at the same horizontal level, the time of flight is equal to $2t_A$.

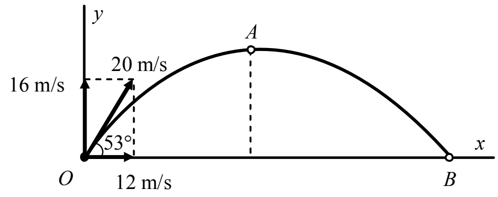


Fig. 4.17

$$T = 2t_A = 2\times 1.6 \text{ s} = 3.2 \text{ s}.$$

❓ Can you find T by using Eq. 4.12?

(b) The projectile lands on the ground at point B, see Fig. 4.17. What are the coordinates of point B?
On multiplying the horizontal component of velocity $v_x (= v_{0x})$ by the time of flight T, we obtain the x-coordinate of the point where the projectile falls.

$$R = x_B = (12 \text{ m/s}) \times (3.2 \text{ s}) = 38.4 \text{ m}.$$

Thus, the coordinates of point B are (38.4, 0) m.
We could have calculated the range of the projectile by directly using the Formula 4.16. But it must be borne in mind that a line of reasoning/approach is more important than a formula.

(c) What is the maximum height above the ground attained by the projectile during the course of its flight?
Formula 4.12 gives the variation of y- coordinate with time. Substituting into this formula $t_A = 1.6$ s for t, we obtain the y- coordinate corresponding to the apex A of the trajectory, which is the height H to which the projectile ascends.

$$H = (16 \text{ m/s})\times(1.6 \text{ s}) - \frac{1}{2}\times(10 \text{ m/s}^2)\times(1.6 \text{ s})^2 = 12.8 \text{ m}.$$

You can verify that Formula 4.14 gives the same result.

(d) At the moment $t = 1$ s, what is the y- coordinate of the projectile?
Substituting $t = 1$s and the values of other quantities into Formula 4.12 you get the y- coordinate of the projectile as

$$y = (16 \text{ m/s})\times(1 \text{ s}) - \frac{1}{2}(10 \text{ m/s}^2)\times(1 \text{ s})^2 = 11 \text{ m}.$$

(e) What is the equation of the path followed by the projectile?
Coordinates of the point where the projectile is at time t are

$$x = 12t$$

and $\quad y = 16t - \frac{1}{2}(10)t^2.$

Substituting $t = \frac{x}{12}$ from the first equation into the second we obtain the equation of the trajectory

$$y = \frac{4}{3}x - \frac{5}{144}x^2.$$

(f) At what moment of time the height of the projectile above the ground is 11 m?
We substitute $y = 11$ m into Eq. 4.12 for obtaining the required time:

$$11 \text{ m} = (16 \text{ m/s})t - \frac{1}{2}(10 \text{ m/s}^2)t^2,$$

which gives $t = 1$ s and 2.2 s.

For both upward and downward motions the acceleration of the projectile is same, the time it takes from 11-m height to the apex is same as the time of motion from the apex to a height of 11 m again. It follows that the points are symmetrical about the apex A, which means that the trajectory is symmetrical about point A, or more specifically, about the vertical line from the apex A.

(g) At the moments the y- component of the displacement of the projectile is 11 m, what is its y- component of velocity?
In the previous problem we computed the moments of time at which the height of the projectile above the ground is 11 m. Using Eq. 4.10 we obtain the velocity of the projectile at these two moments,

at $t = 1$ s, $v_y = 16 \text{ m/s} - 10 \text{ m/s}^2 \times 1 \text{ s} = 6 \text{ m/s}$,

at $t = 2.2$ s, $v_y = 16 \text{ m/s} - 10 \text{ m/s}^2 \times 2.2 \text{ s} = -6 \text{ m/s}$.

The velocity of the projectile when it is at a height of 11 m from the launch point can also be calculated from Eq. 4.6d as follows: set $y = 11$ m, $a_y = -10 \text{ m/s}^2$ and $v_{0y} = 16$ m/s in this equation -

$$v_y^2 = (16 \text{ m/s})^2 - 2(10 \text{ m/s}^2) \times (11 \text{ m}),$$

which yields $v_y = \pm 6$ m/s, as expected. These velocities are shown in Fig. 4.18.

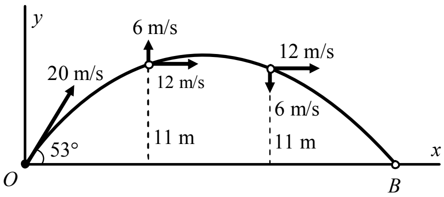
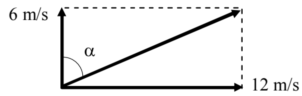


Fig. 4.18

(h) At the moment the y- component of displacement of the projectile is 11 m, what is the angle between its velocity and y- component of velocity?
It suffices to know the two components v_x and v_y of the velocity at a moment (or a position) to compute the angle between the velocity and its y- component. We calculated the y- component of velocity at $y = 11$ m in the previous problem. The required angle can be obtained from the velocity diagram.

$$\tan\alpha = \frac{v_x}{v_y} = \frac{12 \text{ m/s}}{6 \text{ m/s}} = 2,$$

or $\quad \alpha = \tan^{-1} 2.$



(i) At what moment of time the velocity of the projectile is parallel to x- axis?
At the apex of the trajectory velocity has only horizontal component, the vertical component v_y vanishes. The projectile reaches the apex at $t = 1.6$ s; thus, at this moment its velocity is parallel to x- axis.

(j) At what moment(s) of time the x- component of velocity of projectile is twice the y- component in magnitude?
This question was indirectly answered in Parts (g). However, alternatively, we can express the x- and y- components of velocity at moment t as

$$v_x = 12 \text{ m/s and } v_y = (16 - 10t) \text{ m/s}.$$

The x- component of velocity is twice of the y- component in magnitude if

$$12 = 2\,|16 - 10t|$$

$\Rightarrow \quad 12 = 2(16 - 10t)$, which gives $t = 1$ s,

and $\quad 12 = -2(16 - 10t)$, which gives $t = 2.2$ s .

(k) At time t ($t > 0$), the coordinates of the projectile are (x, y). Find whether there is a value of t for which $y = 2x$.
It can be shown that in the interval $0 < t \le 3.2$ s, there is no instant of time at which the y- coordinate is twice as large as x- coordinate. Using simple algebra we can show that the equation $y = 2x$, i.e., $16t - \frac{1}{2}t^2 = 12t$ does not have any real solution in the interval (0, 3.2 s].

(l) At the moment of time t_1 ($t_1 > 0$) the x- component of displacement of the projectile is twice of its y- component of displacement. What is the value of t_1?
Now we are so comfortable with formulas and formulations that the best strategy comes to our mind immediately. To arrive at the answer of this question we simply write

$$12t_1 = 2\left(16t_1 - \frac{1}{2}10t_1^2\right)$$

which gives $t_1 = 2.0$ s.

(m) At the moment the x- component of the displacement of the projectile is twice of its y- component of displacement, what is the angle between the x- component of velocity and velocity?
In the previous problem we calculated the moment of time at which the x- component of the displacement is twice the y- component of the displacement. At this

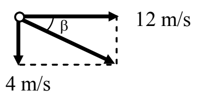
moment of time $(t = 2.0\,\text{s})$, the x- and y- components of the velocity of the projectile are

$v_x = 12$ m/s and $v_y = 16\ \text{m/s} - (10\ \text{m/s}^2) \times (2\ \text{s}) = -4$ m/s,

that is, 4 m/s in downward direction.

Here, the angle that is sought is given by

$$\tan\beta = \frac{4\ \text{m/s}}{12\ \text{m/s}} = \frac{1}{3}$$



or $\qquad \beta = \tan^{-1}\left(\dfrac{1}{3}\right).$

(n) Show that there is no instant of time t ($t > 0$) at which the position vector of the projectile is perpendicular to its velocity vector.

From the two right angle triangles in Fig. 4.19,

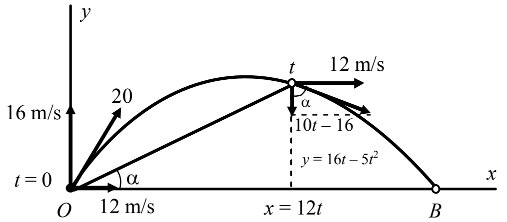


Fig. 4.19

$$\frac{y}{x} = \frac{v_x}{-v_y} \Rightarrow \frac{16t - 5t^2}{12t} = \frac{12}{10t - 16},$$

which simplifies to $5t^2 - 24t + 40 = 0.$

If the velocity vector of the projectile is perpendicular to its position vector at certain instant of time, the scalar product of these two vectors must equal zero at this instant. Figure 4.19 has been constructed under the assumption that the velocity of the projectile is perpendicular to its displacement at an instant t. We have

$$[12\hat{i} + (16 - 10t)\hat{j}] \cdot [12t\,\hat{i} + (16t - 5t^2)\hat{j}] = 0,$$

which yields the same quadratic as above.

Further, if we represent by $\hat{v}_0$ the unit vector in the direction of initial velocity vector, then we can vectorially express the condition $\vec{v} \perp \vec{r}$ as

$$(20\hat{v}_0 + \vec{g}t) \cdot \left(20\hat{v}_0 t + \frac{1}{2}\vec{g}t^2\right) = 0.$$

To evaluate the scalar product in the above equation, note that the angle between unit vector $\hat{v}_0$ and $\vec{g}$ is $90^\circ + \theta = 90^\circ + 53^\circ$. The same quadratic again!

The quadratic equation $5t^2 - 24t + 40 = 0$ does not have any real solution. This implies that during the flight, the velocity of the projectile is never perpendicular to its position vector.

A very fascinating implication of this result is that the projectile always moves away from the thrower. At the moment of projection the projectile has a velocity of 20 m/s directed away from the thrower. As time passes, the velocity of separation decreases (can you show it?), but it never reduces to zero in this case. Throughout the motion the projectile has a component of velocity directed away from the thrower. Consequently, it moves away from him until it hits the ground.

If the velocity vector were to become perpendicular to the position vector at some point in the trajectory, then, in the subsequent motion a component of velocity would be directed inside toward the point of projection, the projectile would be coming closer to the thrower. This situation does not arise in this example. The projectile always moves away from the thrower.

❓ What is the maximum angle to the horizontal at which the projectile can be thrown and always be moving away from the thrower? See Example 6.

(o) During its course of flight the projectile just clears two parallel walls each of height 11 m. What is the separation between the walls?

As computed in Part (f), the projectile is at a height of 11 m at the moments $t = 1$ s and $t = 2.2$ s.

These are the moments at which the projectile clears the 11-m high walls, one after the other, Fig. 4.20. On multiplying v_x by the time interval $\Delta t = 2.2\ \text{s} - 1.0\ \text{s} = 1.2$ s the projectile takes in clearing the walls, we obtain the separation between them

$\Delta x = v_x \Delta t = 12\ \text{m/s} \times 1.2\ \text{s} = 14.4$ m.

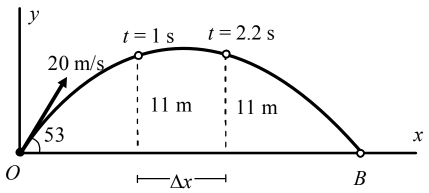


Fig. 4.20

(p) At what moment of time t the velocity of the projectile is perpendicular to its initial velocity?

Figure 4.21 solves this problem immediately.

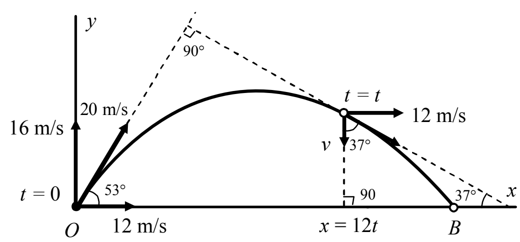


Fig. 4.21

From the triangle of velocities at time t we see that $\tan 37^\circ = \dfrac{v}{12}$, which gives $v = 9$ m/s $\Rightarrow v_y = -9$ m/s, whence $-9\ \text{m/s} = 16\ \text{m/s} - (10\ \text{m/s}^2)t$, giving $t = 2.5$ s.

We can approach this problem in many other ways.

(i) $(12i + (16 - 10t)\hat{j}) \cdot (12\hat{i} + 16\hat{j}) = 0$. Expand the dot product to obtain $t = 25$ s.

(ii) $(20\hat{v}_0 + \vec{g}) \cdot (20\hat{v}_0) = 0$, where $\hat{v}_0$ is the unit vector in the direction of initial velocity. Clearly, the angle between $\hat{v}_0$ and $\vec{g}$ is $90^\circ + 53^\circ$. Expand the dot product for answer.

(iii) The choice of the coordinate axes shown in Fig. 4.22 turns out to be extremely convenient. Let the velocity of the projectile be perpendicular to its initial velocity at instant t. The x- components of velocity at this moment must be equal to zero.

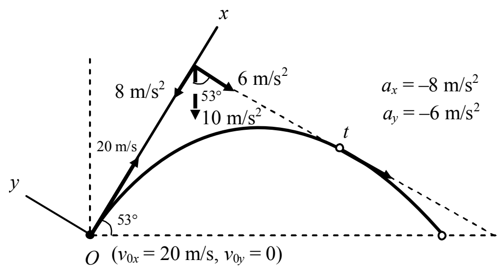


Fig. 4.22

Apply $v_x = u_x + a_x t$ to the motion of the projectile -

$$0 = (20\,\text{m/s}) + (-8\,\text{m/s}^2)t,$$

which gives $t = 2.5$ s.

(q) What is the change in the velocity of the projectile in the time interval $t = 0$ to $t = 1.6$ s?

In case the acceleration of a body is constant, the change in velocity $\Delta\vec{v}$ in time interval Δt is related to the (constant) acceleration $\vec{a}$ as $\Delta\vec{v} = \vec{a}\Delta t$. Hence,

$$\Delta\vec{v} = (-10\hat{j}\ \text{m/s}^2) \times (1.6\ \text{s} - 0) = -16\hat{j}\ \text{m/s}.$$

❓ Use $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ to arrive at the answer.

(r) What is the average acceleration of the projectile in the time interval $t = 0$ to $t = 1.6$ s?

In Section 3.3.2, Chapter 3 we showed that for a uniformly accelerated motion the average acceleration over any interval of time is same and equal to the instantaneous acceleration. Therefore, the average acceleration of the projectile in the time interval $t = 0$ to $t = 1.6$ s is $-10\ \text{m/s}^2\,\hat{j}$.

You can also use the definition of average acceleration and find the answer.

(s) In the time interval $t = 0$ to $t = 1.6$ s, what is the average velocity of the projectile?

For a uniformly accelerated motion the average velocity in *any* time interval is

$$\vec{v}_{av} = \frac{\vec{u} + \vec{v}}{2},$$

where $\vec{u}$ and $\vec{v}$ are the velocities at the beginning and at the end of the time interval. At the initial moment $t = 0$, the velocity of the projectile is $(12\hat{i} + 16\hat{j})$ m/s and at the moment $t = 1.6$ s, its velocity is $12i$ m/s. Substituting these values in the above formula we obtain the average velocity of this time interval,

$$\vec{v}_{av} = \frac{(12\hat{i} + 16\hat{j})\text{m/s} + (12\hat{i})\ \text{m/s}}{2} = (12\hat{i} + 8\hat{j})\ \text{m/s}.$$

We shall now calculate the average velocity using its definition: it is the displacement divided by the time interval in which the displacement was undergone. From the solution of Parts (b) and (c), we can see that the displacement of the projectile in the time interval $t = 0$ to $t = 1.6$ s is $\vec{r} = (19.2\hat{i} + 12.8\hat{j})$ m. (How?) Therefore, the required average velocity is

$$\vec{v}_{av} = \frac{(19.2\hat{i} + 12.8\hat{j})\ \text{m}}{1.6\ \text{s}} = (12\hat{i} + 8\hat{j})\ \text{m/s}.$$

(t) If v_{av} represents the average speed and $|\vec{v}_{av}|$ the magnitude of the average velocity for the time interval $t = 0$ to $t = 1.6$ s, show that $v_{av} > |\vec{v}_{av}|$.

It is obvious that the length of the curve (parabola) is greater than the length r of the chord in Fig. 4.23. Hence the average speed v_{av} is larger than magnitude of average velocity $|\vec{v}_{av}|$. This statement is true for any time interval.

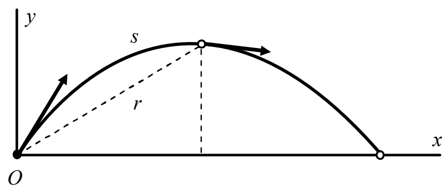


Fig. 4.23

(u) A girl located on x- axis 60 m away from the point of projection starts running at the moment the projectile is thrown. How fast, and in which direction must she run in order to catch the projectile at the level from which it was thrown, if she runs at a constant speed?

We shall use the solution of Parts (a) and (b). To catch the projectile at the same level from which it was thrown the girl must cover a distance of $60\ \text{m} - 38.4\ \text{m} = 21.6\ \text{m}$ in 3.2 seconds. Hence she must run at a speed of $21.6\ \text{m} / 3.2\ \text{s} = 6.75\ \text{m/s}$ towards the point of projection.

(v) A catcher is standing on x- axis 36 m away from the point of projection, see Fig. 4.24. How far above its initial level is the projectile caught?

Projectile reaches the catcher in $36\ \text{m} / (12\ \text{m/s}) = 3$ s. At this moment of time the projectile is at a height of

$$h = (16\ \text{m/s}) \times (3\ \text{s}) - \frac{1}{2} \times (10\ \text{m/s}^2) \times (3\ \text{s})^2 = 3\ \text{m}.$$

Either the catcher is really tall or good at jumping!

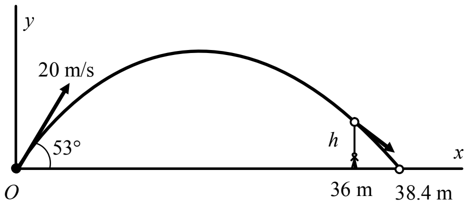


Fig. 4.24

(w) The catcher in Part (v) sees the projectile when it is right above his head and decides to run and catch it before it falls on the ground. If his reaction time is 0.5 s, will he succeed?

No. The projectile comes to the initial level with launch point in $3.2\text{ s}-3\text{ s}=0.2\text{ s}$, whereas the catcher requires 0.5 s to start running.

(x) At what moment of time is the speed of the projectile 15 m/s?

The formula $\sqrt{v_x^2+v_y^2}=v$ gives

$$\sqrt{(12\,\text{m/s})^2+(16\,\text{m/s}-(10\,\text{m/s}^2)\,t)^2}=15,$$

or $\quad t = 0.7$ s, 2.5 s.

(y) At the instant t = 0.7 s, what are the normal and tangential components of the acceleration of the projectile?

The normal component of acceleration is the component that is perpendicular to the velocity vector, while the tangential component is in the direction of velocity vector. You are advised to revisit this problem after going through Section 4.5.1.

At every moment of time during the flight the acceleration of the projectile is in the downward direction and is equal to the acceleration due to gravity g. As shown in Fig. 4.25, the normal and tangential components of the acceleration are $a_N = g\cos\beta$ and $a_\tau = -g\sin\beta$.

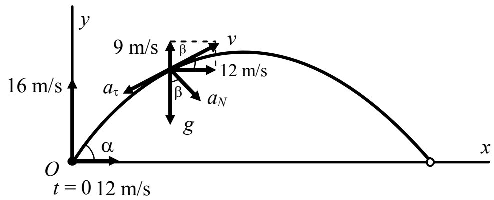


Fig. 4.25

The angle β can be obtained from the velocity diagram. At the moment $t = 0.7$ s the horizontal and vertical components of velocity are $v_x = 12$ m/s and $v_y = 16\text{ m/s} - 10\text{ m/s}^2\times 0.7\text{ s} = 9$ m/s. From the velocity diagram we get $\tan\beta=\frac{3}{4}$, which gives $\beta\approx 37^\circ$. Therefore,

$$a_N = g\cos\beta = 10\text{ m/s}^2\times\frac{4}{5} = 8\text{ m/s}^2$$

and $\quad a_\tau = -g\sin\beta = -10\text{ m/s}^2\times\frac{3}{5} = -6\text{ m/s}^2.$

Pay attention to the fact that the tangential acceleration is negative at the given moment. Does the tangential acceleration of the projectile have a negative value at all positions during its motion?

(z) At what moment of time the magnitude of tangential component of acceleration of the projectile is maximum?

We observe that the angle β in Fig. 4.25 assumes its maximum value at moments $t=0$ and $t=3.2$ s. Hence, obviously, these are the moments at which the tangential acceleration of the projectile will have its maximum value. See solution of Part (y)

At what moment of time the normal component of the acceleration of the projectile has its maximum value?

This happens when the angle β (Fig. 4.25) is the least. At the apex of the trajectory, the velocity of the projectile is perpendicular to its acceleration, the radial acceleration becomes equal to the acceleration due to gravity $g = 10\text{ m/s}^2$.

At what moment(s) of time the magnitude of the normal acceleration is twice the magnitude of the tangential acceleration?

We shall correlate the acceleration diagram to the velocity diagram to obtain the vertical components of the velocity at the moment the given condition is satisfied, see Fig. 4.26. Since $a_N = 2|a_\tau|$ we have $g\cos\beta = \pm\, 2g\sin\beta$ or $\tan\beta = \pm\frac{1}{2}$. The velocity diagram gives

$$v_y = v_x\tan\beta = \pm\ 6\text{ m/s}.$$

Now substituting this value of v_y into Eq. 4.10, we obtain

$$6\text{ m/s} = 16\text{ m/s} + (-10\,\text{m/s}^2)t_1 \text{ or } t_1 = 1\text{ s}$$

and $\quad -6\text{ m/s} = 16\text{ m/s} + (-10\,\text{m/s}^2)t_2$ or $t_2 = 2.2$ s.

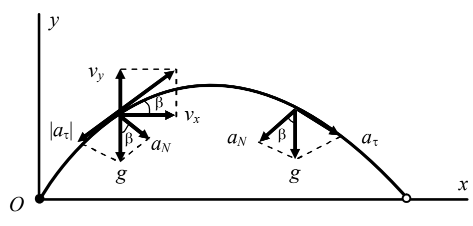


Fig. 4.26

Example 4. A particle is projected from a horizontal x-z plane such that its velocity vector at time t is given by $\vec{v} = a\hat{i} + (b - ct)\hat{j}$. Find (a) time of flight T; (b) maximum height H; (c) range R.

(a) The time of flight is the time taken for the vertical displacement to become zero. We have

$$v_y = b - ct$$

or $$\frac{dy}{dt} = b - ct$$

or $$y = \int_0^t (b - ct)dt.$$

If y becomes zero again after time T,

$$0 = bT - \frac{cT^2}{2},$$

which gives $T = \frac{2b}{c}$.

(The solution $T = 0$ corresponds to instant of projection.)

Alternatively, the vertical velocity becomes zero at time given by $b - ct = 0$ or $t = \frac{b}{c}$, and time of flight is twice this time, (why?), hence $T = \frac{2b}{c}$.

(b) The maximum height is the displacement in y- direction at the moment the y- component of velocity is zero.

We know that v_y is zero at $t = \frac{b}{c}$. So the maximum height is

$$H = \int_0^{b/c} (b - ct)dt = b\left(\frac{b}{c}\right) - \frac{c}{2}\left(\frac{b}{c}\right)^2 = \frac{b^2}{2c}.$$

Alternatively, the motion of particle is one with constant acceleration. (Can you see this?) Application of Eq. 4.6d gives

$$v_y^2 - u_y^2 = 2a_y H$$

or $$0^2 - b^2 = 2(-c)H$$

or $$H = \frac{b^2}{2c}.$$

Further, the displacement in the vertical direction in time t is

$$y = \int_0^t (b - ct)\,dt = bt - \frac{ct^2}{2}$$

$$= -\frac{c}{2}\left[t^2 - \frac{2b}{c}t\right]$$

$$= -\frac{c}{2}\left[t^2 - 2t\left(\frac{b}{c}\right) + \left(\frac{b}{c}\right)^2 - \left(\frac{b}{c}\right)^2\right]$$

$$= -\frac{c}{2}\left[\left(t - \frac{b}{c}\right)^2 - \left(\frac{b}{c}\right)^2\right]$$

$$= \frac{b^2}{2c} - \frac{c}{2}\left(t - \frac{b}{c}\right)^2.$$

Clearly, for y to be maximum second term in the above expression must be zero. Hence $H = \frac{b^2}{2c}$.

(c) Range is the horizontal displacement in the time of flight. Hence,

$$R = \int_0^{2b/c} v_x dt = \int_0^{2b/c} a dt = \frac{2ab}{c}.$$

An alternative solution to the whole problem: compare the given velocity $\vec{v} = a\hat{i} + (b - ct)\hat{j}$ with the velocity of a simple familiar projectile

$$\vec{v} = v_0 \cos\theta \hat{i} + (v_0 \sin\theta - gt)\hat{j}$$

term by term -

$$v_0 \cos\theta = a,\ v_0 \sin\theta = b \text{ and } g = c.$$

The problem henceforth reduces to the computation of T, H and R in projectile motion.

Example 5. As shown in Fig. 4.27, a particle is projected from point A at angle β to the horizontal where $\beta > \alpha$. Find the speed of projection of the particle so that it just grazes the inclined wall?

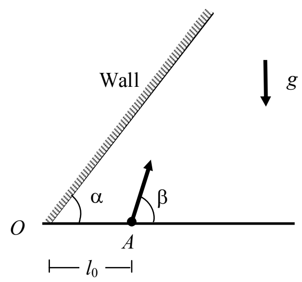


Fig. 4.27

Let the particle be projectile with speed v_0. At the moment the particle grazes the wall its velocity vector is parallel to it (why?). Let x_0 and y_0 be the horizontal and vertical displacements of the particle at that instant.

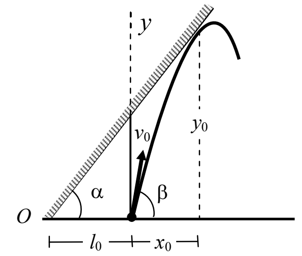


Fig. 4.28

From Fig. 4.28,

$$\tan\alpha = \frac{y_0}{l_0 + x_0}. \qquad \ldots\text{(i)}$$

Also, the velocity of the particle at time t is

$$\vec{v} = (v_0\cos\beta)\hat{i} + (v_0\sin\beta - gt)\hat{j}.$$

The time when the velocity vector makes angle α with the horizontal is given by

$$\tan\alpha = \frac{v_0\sin\beta - gt}{v_0\cos\beta}$$

which gives $t = \dfrac{v_0\sin(\beta-\alpha)}{g\cos\alpha}$.

At this time,

$$x_0 = (v_0\cos\beta)t = \frac{v_0^2\sin(\beta-\alpha)\cos\beta}{g\cos\alpha},$$

and $\quad y_0 = (v_0\sin\beta)t - \frac{1}{2}gt^2$

$$= \frac{v_0^2\sin\beta\sin(\beta-\alpha)}{g\cos\alpha} - \frac{v_0^2\sin^2(\beta-\alpha)}{2g\cos^2\alpha}.$$

Substituting x_0 and y_0 into Eq. (i) we get

$$\tan\alpha = \frac{\dfrac{v_0^2\sin\beta\sin(\beta-\alpha)}{g\cos\alpha} - \dfrac{v_0^2\sin^2(\beta-\alpha)}{2g\cos^2\alpha}}{l_0 + \dfrac{v_0^2\sin(\beta-\alpha)\cos\beta}{g\cos\alpha}},$$

which, after some simple trigonometry, gives

$$v_0 = \sqrt{\frac{2gl_0\sin\alpha\cos\alpha}{\sin^2(\beta-\alpha)}}.$$

Alternatively, the equation of trajectory of the particle in the coordinate system shown in Fig. 4.28 is

$$y = x\tan\beta - \frac{gx^2}{2v_0^2\cos^2\beta}$$

and that of the wall projected onto the plane of the trajectory of the projectile is

$y = (\tan\alpha)(x + l_0)$.

If the particle just grazes the wall, then the points of intersection of the above two curves must coincide. If these equations have two distinct solutions, the particle hits the wall; no real solution implies that the particle misses the wall.

On eliminating y from the two equations we get

$$\left(\frac{g}{2v_0^2\cos^2\beta}\right)x^2 + x(\tan\alpha - \tan\beta) + l_0\tan\alpha = 0.$$

The roots of the above equation coincide if

$$(\tan\alpha - \tan\beta)^2 = \frac{4g}{2v_0^2\cos^2\beta}l_0\tan\alpha$$

or $\quad \dfrac{\sin^2(\beta-\alpha)}{\cos^2\alpha\cos^2\beta} = \dfrac{2gl_0\tan\alpha}{v_0^2\cos^2\beta}$

or $\quad v_0 = \sqrt{\dfrac{2gl_0\sin\alpha\cos\alpha}{\sin^2(\beta-\alpha)}}.$

In this method the condition $\beta > \alpha$ is not needed. Can you reason it out geometrically?

Here is the smartest way to solve this problem. Choosing a coordinate system wisely does wonders at times.

We choose a coordinate system as shown in Fig. 4.29. Clearly, $l_0\sin\alpha$ is the maximum y- displacement if the particle grazes wall.

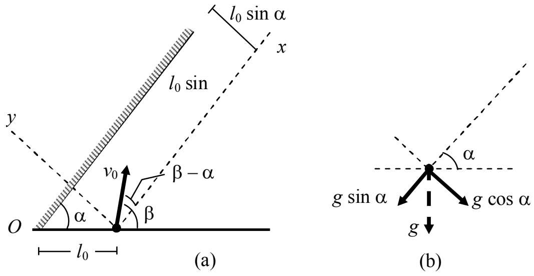


Fig. 4.29

Substituting $v_{0y} = v_0\sin(\beta-\alpha)$, $v_y = 0$, $a_y = -g\cos\alpha$, and $y_0 = l_0\sin\alpha$ into Eq. 4.6d we obtain

$$0 - v_0^2\sin^2(\beta-\alpha) = 2(-g\cos\alpha)l_0\sin\alpha,$$

which gives $v_0 = \sqrt{\dfrac{2gl_0\sin\alpha\cos\alpha}{\sin^2(\beta-\alpha)}}.$

Example 6. What is the maximum angle to the horizontal at which a stone can be thrown and always be moving away from the thrower?

If the stone is projected at a large angle to the horizontal, say 80° or so, it can be intuitively figured out or can be inferred by a look at the trajectory (Fig. 4.30) that its distance from the thrower will first increase and then decrease, attaining a maximum value in between.

For some time the stone moves away from the point of projection, velocity component on the line connecting the point of projection to instantaneous position of the stone is directed away from the thrower as shown in the figure. It can be further seen that with the passing of time the velocity of separation decreases. At a certain moment of time the velocity of separation becomes zero. This happens at the moment when the velocity of the stone is perpendicular to its position vector. Thereafter, the velocity component projected on the position vector is directed towards the point of projection; the distance of the stone from the thrower decreases.

The above argument immediately gives us a way to solve the problem: the angle of projection must be such that the velocity of the stone is never perpendicular to its position vector during its flight, that is, the stone must always have a velocity of separation as seen by the thrower.

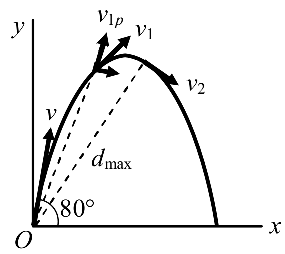


Fig. 4.30

As in Example 3 (Part **n**), to start with develop the equation that gives the moment of time t at which the velocity of the stone is perpendicular to its position vector. From this equation find the values of θ for which it has no real solution. That's it!

We can develop the required equation in the following ways.

(i) Figure 4.31 shows the velocity and position vectors of the stone at a certain instant t. If $\vec{v}_t$ is perpendicular to $\vec{r}_t$, we have $\vec{r}_t \cdot \vec{v}_t = 0,$ which can be written as

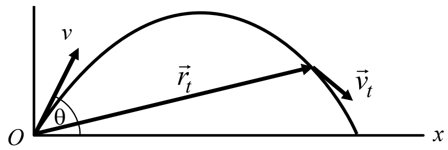


Fig. 4.31

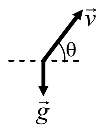
$$\left((v\cos\theta)t\hat{i} + \left((v\sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}\right)\cdot\left(v\cos\theta\hat{i} + (v\sin\theta - gt)\hat{j}\right) = 0$$

which gives

$$t^2 - \left(\frac{3v\sin\theta}{g}\right)t + \frac{2v^2}{g^2} = 0.$$

(ii) We can express the scalar product of the position and velocity vectors without resorting to components and proceed straightaway as below.

We have $\vec{r}_t \cdot \vec{v}_t = 0$



or $\quad \left(\vec{v}t + \frac{1}{2}\vec{g}t^2\right)\cdot(\vec{v} + \vec{g}t) = 0$

or $\; v^2 t + vgt^2\cos(90^\circ + \theta) + \frac{1}{2}vgt^2\cos(90^\circ + \theta) + \frac{1}{2}g^2t^3 = 0$

which gives

$$t^2 - \left(\frac{3v\sin\theta}{g}\right)t + \frac{2v^2}{g^2} = 0.$$

(iii) The motion of the stone is given by the following equations (Fig. 4.32):

$$x = vt\cos\theta, \quad y = vt\sin\theta - \frac{g}{2}t^2,$$

$$v_x = v\cos\theta, \quad v_y = v\sin\theta - gt, = -v \text{ (say)}.$$

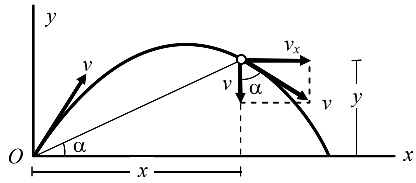


Fig. 4.32

It can be seen from the figure that at the instant the velocity of the stone is perpendicular to its position vector, $\frac{y}{x} = \frac{v_x}{-v_y} = \frac{v_x}{v}$, which yields the equation

$$t^2 - \frac{3v\sin\theta}{g}t + \frac{2v^2}{g^2} = 0.$$

If the velocity of the stone is never perpendicular to its position vector, the discriminant of the quadratic equation in t must be negative, that is,

$$\left(\frac{3v\sin\theta}{g}\right)^2 < 8\left(\frac{v^2}{g^2}\right).$$

Thus, for the stone to be always moving away from the thrower, we must have $\sin\theta < \sqrt{8/9} = 0.94$, i.e., $\theta < 70.5°$. ◀

4.4.2. Motion of a Body Thrown along the Horizontal

Let us consider the motion of a body thrown along the horizontal and moving only under the action of the force of gravity. We shall again ignore the air resistance. Let us, for example, throw a particle from a tower, its initial velocity v_0 being directed along the horizontal, Fig. 4.33(a).

We shall analyze the motion of the particle onto the (downward) vertical y- axis and the horizontal x- axis. The motion of the particle on the x- axis is the motion with zero acceleration at a velocity $v_x = v_0$. The motion in the y- direction is a free fall with an acceleration $a_y = g$ under the action of the force of gravity with zero initial velocity.

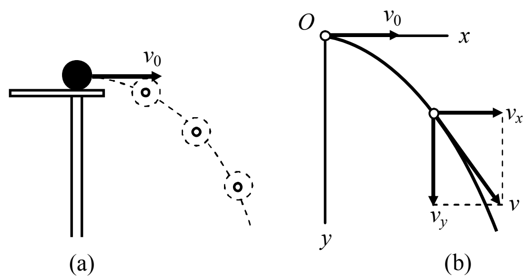


Fig. 4.33

The v_x component remains constant and equal to v_0. The v_y component varies with time as: $v_y = gt$. The resultant velocity can be easily found with the help of the parallelogram rule as shown in Fig. 4.33(b).

In the coordinate system shown in the figure on the assumption that it was projected from the origin at $t = 0$, the coordinates of the particle at a moment t will be

$$x = v_0 t,$$

and $$y = \frac{1}{2} g t^2.$$

In order to find the equation of the trajectory, we express t in the first equation in terms of x and substitute it into the second. This gives

$$y = \frac{g}{2v_0^2} x^2.$$

The graph of this function is shown in Fig. 4.33(b). Such trajectories are called parabolas. Thus, *a freely falling body with an initial horizontal velocity moves along a parabola.*

The distance covered in the vertical direction does not depend on the initial velocity. However, the distance covered in the horizontal direction is proportional to the initial velocity. Therefore, at a high initial horizontal velocity, the parabola along which the particle falls is stretched in the horizontal direction.

If we know the initial velocity v_0 and the height h from which the body is thrown, we can calculate the horizontal range R to the place where the particle falls. On substituting $y = h$ and $x = R$, in the above equation we obtain

$$R = v_0 \sqrt{\frac{2h}{g}}.$$

❸ A ball rolling off a table of 1 m height lands at a distance of 2 m from the edge of the table. What was the horizontal velocity of the ball? Neglect air resistance. Take $g = 10 \text{ m/s}^2$.

▶ **Example 7.** Assume that a projectile is launched from a tower of height 16 m with a velocity 20 m/s making an angle of $53°$ with the horizontal, see Fig. 4.34.

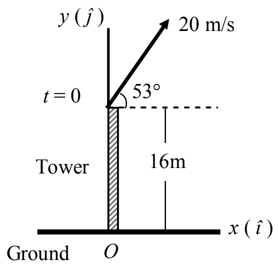


Fig. 4.34

(a) How long does the projectile take to hit the ground?

Substituting $v_{0y} = 16 \text{ m/s}$, $a_y = -10 \text{ m/s}^2$ and $S_y = -16 \text{ m}$, into Eq. 4.6b we obtain

$$-16 \text{ m} = (16 \text{ m/s})t - \frac{1}{2}\left(10 \text{ m/s}^2\right) t^2$$

which gives $t = 4$ s. The above equation yields a negative root also. Can any physical interpretation be assigned to this root?

(b) How far from the base of the tower does the projectile hit the ground?

On multiplying v_x and the time of flight we obtain the x-coordinate of the point where the projectile falls on the ground: $x = 12 \text{ m/s} \times 12 \text{ s} = 48 \text{ m}$.

(c) What is the maximum height above ground attained by the projectile?

Analyzing the y- component of motion we obtain

$$h_{\max} = h + \frac{v_{0y}^2}{2g} = 16 \text{ m} + \frac{(16 \text{ m/s})^2}{2 \times 10 \text{ m/s}^2} = 28.8 \text{ m}.$$

(d) With what velocity the projectile lands on the ground?

The x- component of velocity remains constant and equal to its initial value $v_0 \cos\theta = 12$ m/s, because the x- component of acceleration is absent. The y- component of velocity varies according to the Formula 4.10. Substituting $t = 4$ s into this formula we obtain $v_y = 16 \text{ m/s} - 10 \text{ m/s}^2 \times 4 \text{ s} = -24 \text{ m/s}$. The projectile lands on the ground with velocity $\vec{v} = (12\hat{i} - 24\hat{j})$ m/s.

(e) The projectile just crosses a wall of height 8.8 m. How sooner after the launch the projectile does this?

When the projectile is just above the wall, its displacement in y- direction is $-(16 \text{ m} - 8.8 \text{ m}) = -7.2 \text{ m}$. Using Formula 4.12 we obtain $-7.2 \text{ m} = (16 \text{ m/s})t - \frac{1}{2} \times 10 \text{ m/s}^2 \times t^2$, which gives $t = 3.6$ s. Why did we neglect the other root of this equation?

(f) With what velocity the projectile crosses the wall in Part (e)?

$$\vec{v} = v_x\hat{i} + v_y\hat{j} = v_{0x}\hat{i} + (v_{0y} - gt)\hat{j}$$
$$= 12\hat{i} \text{ m/s} + (16 \text{ m/s} - 10 \text{ m/s}^2 \times 3.6 \text{ s})\hat{j}$$
$$= (12\hat{i} - 20\hat{j}) \text{ m/s}.$$

(g) What is the horizontal distance between the wall and tower in Part (e)?

$$x_{\text{wall}} = v_{0x} \times t = 12 \text{ m/s} \times 3.6 \text{ s} = 43.2 \text{ m}.$$

(h) What is the equation of the path of the projectile in the coordinate system given in the figure?

The x- and y- coordinates of the projectile vary with time according to the laws (Fig. 4.35):

$$x = 12t$$

$$y = 16 + 16t - 5t^2.$$

In order to find the equation of the trajectory, we substitute $t = \dfrac{x}{12}$ from the first equation into the second. This gives

$$y = 16 + 16\left(\frac{x}{12}\right) - 5\left(\frac{x}{12}\right)^2.$$

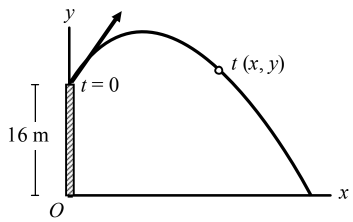


Fig. 4.35

(i) At what moment of time the speed of projectile is 24 m/s?

Using $v = \sqrt{v_x^2 + v_y^2}$ we can write

$$\sqrt{(12\text{ m/s})^2 + ((16 - 10\,t)\text{m/s})^2} = 24\text{ m/s}$$

which yields $t = 3.68$ s.

Example 8. A cannon situated on the top of a hill of height 3000 m fires two shots, each with the same speed $100\sqrt{3}$ m/s at some interval of time, one shot upwards at an angle of 60° with the horizontal and the other shot horizontally. The shots collide in air at point P. Take g = 10 m/s^2. Find

(a) the time interval between the firings;

(b) the coordinates of the point P. Take origin of the coordinate system at the foot of the hill right below the cannon.

You can put forward many arguments to show that the shot fired at the angle of 60° to the horizontal takes a longer time to reach point P (you must do it), therefore, it was fired first.

Let the first shot be fired at $t = 0$, and the second shot, fired in the horizontal direction, at moment $t = T$, and the two shots collide at point P at moment $t = t$.

Figure 4.36 sketches the motion of the shots. On equating the horizontal and vertical displacements of the two shots, since they reach point P simultaneously, we get

$$(50\sqrt{3}\text{ m/s})t = (100\sqrt{3}\text{ m/s})(t - T)$$

and $$(150\text{ m/s})t - \frac{1}{2}(10\text{ m/s}^2)t^2 = 0 \times (t - T) - \frac{1}{2}(10\text{ m/s}^2)(t - T)^2.$$

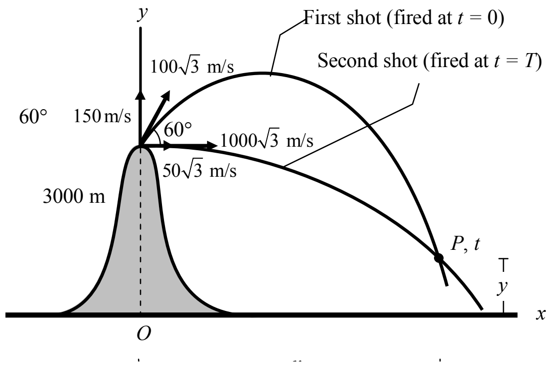


Fig. 4.36

On solving these two equations we obtain $T = 20$ s, and $t = 40$ s. This implies that the first shot travels for time $t = 40$ s before hitting the other shot. Displacement of the first shot in the x- and y- directions in this time are

$$S_x = 50\sqrt{3}\text{ m/s} \times 40\text{ s} = 2000\sqrt{3}\text{ m} = 2\sqrt{3}\text{ km}$$

and $$S_y = (150\text{ m/s}) \times (40\text{ s}) - \frac{1}{2} \times (10\text{ m/s})^2 \times (40\text{ s})^2$$

$$= -2000\text{ m} = -2\text{ km}.$$

Pay attention to the coordinate system given in the problem. The y- component of the displacement of shot *1* from the cannon is –2 km, that means 2 km downward form the point of projection; the y- coordinate of point P is $3\text{ km} - 2\text{ km} = 1\text{ km}$.

The coordinate of point P are $(2\sqrt{3}, 1)$ km.

Alternatively, without delving on which shot was fired first, you can proceed by simply assuming their times of motion until they collide. Assume the time of motion of the first shot is t_1 and that of the second shot is t_2.

This immediately gives

$$(50\sqrt{3}\text{ m/s})t_1 = (100\sqrt{3}\text{ m/s})t_2$$

and $$(150\text{ m/s})t_1 - \frac{1}{2}(10\text{ m/s}^2)t_1^2 = 0 \cdot t_2 - \frac{1}{2}(10\text{ m/s}^2)t_2^2$$

which give $t_1 = 40$ s and $t_2 = 20$ s. Now you can proceed further.

❸ (i) What are the equations of the trajectories of the two shots in the coordinate system given in the problem?

(ii) Can a time interval between the firings ensure that the two shots hit each other at the point which is at the same horizontal level as the foot of hill? If yes, what is it equal to?

Example 9. A man stands on the top a tower and throws a ball at a speed of 10 m/s at an angle θ to the horizontal. The height of the tower is 10 m and the ball strikes the ground at a distance of d from the foot of the tower. Find the value of θ for which the distance d is a maximum. Take $g = 10$ m/s^2.

Take a coordinate system whose origin coincides with the base of the tower, x- axis is horizontal in the plane of the trajectory of the ball and y- axis is vertically upwards. (Fig. 4.37).

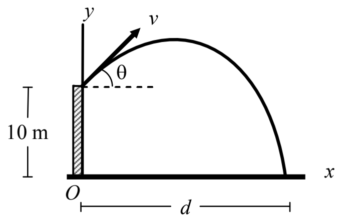


Fig. 4.37

If the ball strikes the ground in time t,

$$-10\,\text{m} = (10\sin\theta)\,\text{m/s} \times t - \frac{1}{2} \times 10\,\text{m/s}^2 \times t^2$$

which gives $t = \sin\theta + \sqrt{2+\sin^2\theta}$.

Distance of the point where the ball strikes from the base of the tower where is

$d = (v\cos\theta)t$

$= (10\cos\theta) \times (\sin + \sqrt{2+\sin^2\theta})$.

On differentiation of the expression for d and some rearrangement we obtain

$$\frac{dd}{d\theta} = 10\left(\sin\theta + \sqrt{2+\sin^2\theta}\right)\left\{-\sin\theta + \frac{\cos^2\theta}{\sqrt{2+\sin^2\theta}}\right\}.$$

The condition $\frac{dd}{d\theta} = 0$ gives the value of θ at which d is maximum. For $0 < \theta < \frac{\pi}{2}$, $(\sin\theta + \sqrt{2+\sin^2\theta}) > 0$, therefore, we have

$$-\sin\theta + \frac{\cos^2\theta}{\sqrt{2+\sin^2\theta}} = 0,$$

which on simplification gives

$$\sin\theta = \frac{1}{2} \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

or θ = 30°.

It is left to you to perform the second derivative test (or otherwise) to ascertain that at θ = 30°, the value of d is maximum. We substitute $\theta = 30^\circ$ into the expression for d to obtain

$$d_m = 10\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} + \sqrt{2+\frac{1}{4}}\right) = 10\sqrt{3}\ \text{m}.$$

For the beginners who are not introduced to calculus yet, here is a graphical method to arrive at the answer. At this level we do not apply such methods for problem solving. However, it is important to be familiar with such alternative methods. Let us project the ball with speed 10 m/s at different angles from the horizontal, and compute the horizontal distance of the point where it lands on the ground from the base of the tower each time. The results of this experiment are complied in Table 4.1

Table 4.1: Dependence of d on θ

θ, degrees	t, s	d, m
0	1.41	14.1
15	1.70	16.4
30	202.	17.3
45	2.30	16.1
60	2.50	12.5
90	2.70	0

Next we construct the d-θ graph, plotting θ and d on two mutually perpendicular axes, Fig. 4.38.

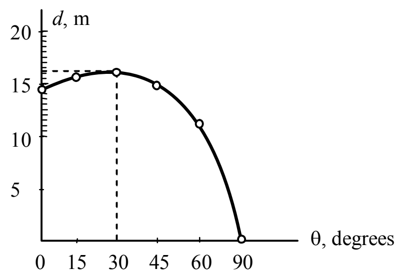


Fig. 4.38

From the graph it can be seen that as the angle of projection θ increases, the distance d of the point where the projectile falls from the base of the tower first increases, attains a maximum value at a certain angle of projection and then decreases, becoming zero at $\theta = 90^\circ$. When θ = 90°, the projectile moves in vertical direction. It can be observed from the graph that d attains its maximum value $d_m = 17.3$ m at the angle of projection $\theta = 30^\circ$.

These solutions involve lengthy cumbersome calculations. But one must be able to perform them when asked to. You must know the steps to be followed and the calculations involved in graphical method. This will help you in your practical classes. However, for such seemingly tedious problems we have fortunately some short elegant methods employing mathematics in ingenious ways. See the solution of this problem in Example 35. ◀

4.4.3. Projectile Motion on an Inclined Plane

A projectile is fired up an incline with an initial speed v_0 at an angle θ to the horizontal as shown in Fig. 4.39. The angle of the incline is α to the horizontal.

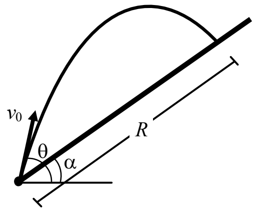


Fig. 4.39

As in Section 4.4, we choose the coordinate axes so that the x- axis is horizontal and y- axis lies in the same vertical plane with the initial velocity v_0. The x- component of the initial velocity is $v_0 \cos\theta$ while the y- component is $v_0 \sin\theta$ (for the directions of x- and y- axes shown in Fig. 4.40). Since the x- component of acceleration is absent, the x- component of velocity remains constant and equal to the initial velocity $v_0 \cos\theta$.

The y- component of motion occurs under the acceleration due to gravity g.

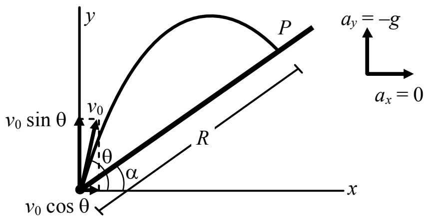


Fig. 4.40

If the distance of the point P where the projectile falls on the incline from the point of projection is R, the x- and y- coordinates of point P are $R\cos\alpha$ and $R\sin\alpha$ respectively. If the projectile takes a time T to hit the incline plane, the coordinates of point of impact are obtained from equations:

$$x = R\cos\alpha = (v_0 \cos\theta)T$$

$$y = R\sin\alpha = (v_0 \sin\theta)T - \frac{1}{2}gT^2.$$

Substituting for T the from the first equation into the second and simplifying, we obtain

$$R = \frac{2v_0^2 \sin(\theta-\alpha)\cos\theta}{g\cos^2\alpha}.$$

It can be seen that for a fixed value of v_0, the expression for R assumes its maximum value when the function $f = \sin(\theta-\alpha)\cos\theta$ is maximum. Using the first principal of maxima-minima or otherwise it can be shown that the function f has its maximum value when $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$. Consequently, the maximum value of R is obtained when the angle of projection $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$.

We substitute $\frac{\pi}{4} + \frac{\alpha}{2}$ for θ into the expression for R to obtain its maximum value:

$$R_{\max} = \frac{v_0^2}{g(1+\sin\alpha)}.$$

If the angle of projection is measured from the incline plane, not from the horizontal, then x- and y- components of the initial velocity are $v_0 \cos(\alpha+\theta)$ and $v_0 \sin(\alpha+\theta)$. Except this fact, every step in the above analysis remains in force, yielding a different expression for R, of course. Do you expect a different expression for $R_{\max}$ also?

Alternatively, we choose the coordinate axes so that x- axis lies in the incline plane along the line of maximum slope, in the same vertical plane with the initial velocity v_0, and y- axis perpendicular to the chosen x- axis, see Fig. 4.41.

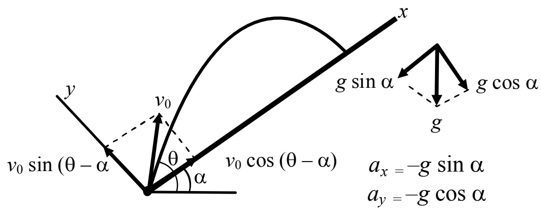


Fig. 4.41

The x- component of initial velocity is $v_0 \cos(\theta-\alpha)$ and the y- component is $v_0 \sin(\theta-\alpha)$. The acceleration of the projectile is g in vertically downward direction. Resolution of g gives the components of the acceleration $a_x = -g\sin\alpha$, $a_y = -g\cos\alpha$.

If the projectile falls on the incline at a point whose coordinates are $(R, 0)$, we have

$$R = v_0 \cos(\theta-\alpha)T - \frac{1}{2}(g\sin\alpha)T^2$$

and $\quad 0 = v_0 \sin(\theta-\alpha)T - \frac{1}{2}(g\cos\alpha)T^2.$

Substituting for T from the second equation into the first we get, after some simple trigonometry,

$$R = \frac{2v_0^2 \sin(\theta-\alpha)\cos\theta}{g\cos^2\alpha}.$$

This approach is distinctly advantageous as compared to the previous one. Mathematical manipulations are a great deal simpler. If you measure the angle of projection from the inclined plane, instead of from the horizontal, you can further reduce the mathematics. We present the bare essential steps leaving the required manipulations to you

$$0 = (v_0 \sin\theta)T - \frac{1}{2}(g\cos\alpha)T^2$$

$$R = (v_0 \cos\theta)T - \frac{1}{2}(g\sin\alpha)T^2.$$

Substituting for T from the first equation into the second we get

$$R = \frac{2v_0^2 \sin\theta\cos(\alpha+\theta)}{g\cos^2\alpha}.$$

Now assume $\sin\theta\cos(\alpha+\theta) = f$ and set $\frac{df}{d\theta} = 0$. You get $\theta = \frac{\pi}{4} - \frac{\alpha}{2}$. Hence

$$R_{\max} = \frac{v_0^2}{g(1+\sin\alpha)}.$$

❓ Demonstrate that if the projectile is fired with initial velocity v_0 down the incline, then the maximum range is

$$R_{\max} = \frac{v_0^2}{g(1-\sin\alpha)}.$$

▶ **Example 10.** A particle is thrown horizontally with a speed v from a point on a plane inclined at an angle α to the horizontal. The trajectory of the particle lies in the vertical plane that contains the line of maximum slope on the incline. (i) How far from the point of projection will the particle land on the plane? (ii) How long does the particle take to do it? (iii) What is the velocity of the particle at the moment it hits the plane?

The choice of coordinate system in this problem is more than obvious. Choose the x- axis in the horizontal direction and y- axis in the vertically downward direction, (Fig. 4.42).

The components of acceleration along the chosen coordinate axes are $a_x = 0, a_y = g$. Also,

at $t = 0,\ x = 0, y = 0, u_x = v, v_y = 0,$

and at $t = t,\ x = d\cos\alpha,\ y = d\sin\alpha$.

(i) For the motion in x- and y- directions we obtain

$$d\cos\alpha = v\,t$$

and $$d\sin\alpha = 0 \times t + \frac{1}{2}g\,t^2.$$

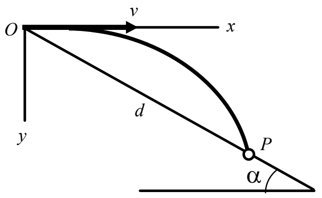


Fig. 4.42

Substituting for t from the first equation into the second and solving for d we obtain

$$d = \left(\frac{2v^2}{g}\right)\tan\alpha\sec\alpha.$$

(ii) The time taken by the projectile to hit the plane

$$t = \frac{d\cos\alpha}{v}$$

$$= \frac{\left(\frac{2v^2\tan\alpha\sec\alpha}{g}\right)\cos\alpha}{v} = \left(\frac{2v}{g}\right)\tan\alpha.$$

We can also find t by dividing the second equation developed in Part(i) by the first

$$\frac{d\sin\alpha}{d\cos\alpha} = \frac{\frac{1}{2}gt^2}{vt}$$

or $$\tan\alpha = \frac{gt}{2v}$$

or $$t = \left(\frac{2v}{g}\right)\tan\alpha.$$

(iii) The y- component of the velocity of the projectile at the moment it hits the plane is

$$v_y = 0 + g\,t = g \times \left(\frac{2v}{g}\right)\tan\alpha = 2v\tan\alpha.$$

Required velocity

$$v' = \sqrt{v_x^2 + v_y^2} = \sqrt{v^2 + (2v\tan\alpha)^2} = v\sqrt{1+4\tan^2\alpha}.$$

The velocity makes an angle of

$$\beta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}(2\tan\alpha)$$ from the horizontal.

Example 11. A small, elastic ball is dropped vertically onto a long plane inclined at an angle α to the horizontal. Is it true that the distances between consecutive bouncing points grow as in an arithmetical progression? Assume that collisions are perfectly elastic and that air resistance can be neglected.

You will learn about collisions in Chapter 9. Here, it suffices to know that on impact, the component of velocity of the ball that is perpendicular to the plane gets reversed, and the component along the plane remains unchanged.

We know how to calculate the displacement undergone in the nth second in rectilinear motion with constant acceleration. This problem is no different. After a little rearrangement the problem can be reduced to a simpler problem as depicted in Fig. 4.43.

It can be immediately seen that the time interval between any two successive impacts has be to the same, $T = \frac{2u_\perp}{g\cos\alpha}$. The distances between the consecutive bouncing points can be easily calculated as under:

$$S_1 = uT + \frac{1}{2}g\sin\alpha(T^2)$$

$$S_2 = u(2T) + \frac{1}{2} g \sin\alpha\,(2T)^2 - \left(uT + \frac{1}{2} \sin\alpha T^2 \right)$$

$$= uT + \frac{1}{2} g \sin\alpha\ \ (3T^2)$$

$$S_3 = u(3T) + \frac{1}{2} g \sin\alpha\,(3T)^2 - \left(u(2T) + \frac{1}{2} g \sin\alpha\,(2T)^2 \right)$$

$$= uT + \frac{1}{2} g \sin\alpha\ \ (5T^2)$$

$$S_4 = u(4T) + \frac{1}{2} g \sin\alpha (4T)^2 - \left(u(3T) + \frac{1}{2} g \sin\alpha (3T)^2 \right)$$

$$= uT + \frac{1}{2} g \sin\alpha\ \ (7T^2).$$

It is obvious that $S_1, S_2, S_3, ...$ form an arithmetical progression.

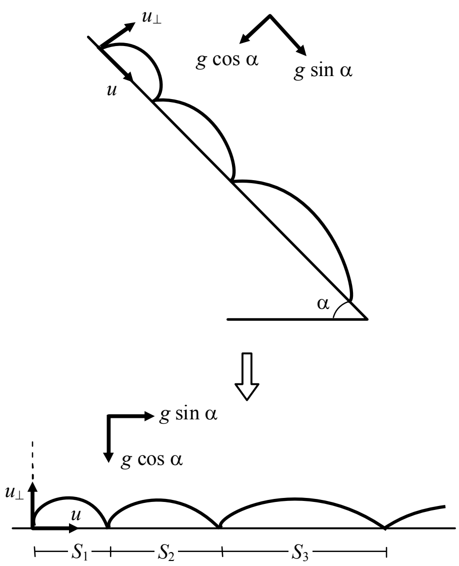


Fig. 4.43

You can also compute the displacements in time nT and in time $(n - 1)T$. Then compute $S_{(nT)\text{th}}$, and show that S's are in arithmetical progression.

Alternatively, Fig. 4.44 shows the velocity-time graph of a motion with constant acceleration. Let the displacement undergone in the time interval $(4T - 3T)$ be denoted by S and that in the interval $(5T - 4T)$ by S'.

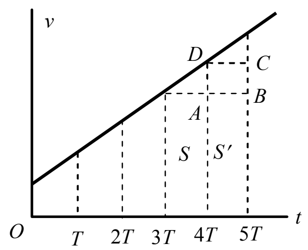


Fig. 4.44

The difference $S' - S$ is equal to the area of the rectangle $ABCD$. This is all that you need to conclude that the distances between the consecutive bounces in the problem are in arithmetical progression.

Further, you can also argue intuitively that the motion in the direction perpendicular to the plane consists of bounces of identical heights, i.e., of identical periods. The component of the take off velocity perpendicular to the plane is same after each bounce, and also the acceleration component perpendicular to the plane is same $g \cos\alpha$ throughout the motion. And, since the acceleration along the plane is constant, the ball's average speed between bounces *increases uniformly*, and so the distances between two consecutive bounces increase in an arithmetical progression. ◀

4.4.4. Flight of Bullets

When the velocities of projectiles are high, air resistance considerably alters their motion in comparison with the results of calculations carried out in the previous sections. If air resistance were absent, the maximum range of the projectile would be observed, as was mentioned in Section 4.4.1, at the angle of projection equal to 45°. The effect of air resistance on the flight of projectiles becomes weaker for larger projectiles for the reason the mass of a projectile increases as the cube of its size, while the force of air resistance increases as the square of its size. Thus, the ratio of air resistance to the mass of the projectile, i.e., the effect of air resistance, decreases with the increasing size. Therefore, for the same initial velocity of projectiles fired from a gun, their range increases with the calibre.

It can be shown that air resistance leads to a change in the trajectory of a bullet such that the angle of inclination corresponding to the maximum range turns out to be less than 45° (it is different for different initial velocities of the bullet). At the same time, the horizontal range (as well as the maximum height of the flight) turns out to be much smaller. For example, for an initial velocity of 800 m/s and angle of 45°, the horizontal range of the bullet is $\frac{(800 \text{ m/s})^2 \sin 90^\circ}{10 \text{ m/s}^2} = 64000 \text{ m} = 64 \text{ km}$ in the absence of the resistance of the medium and assuming that acceleration due to gravity in constant 10 m/s^2 at each point of the trajectory. However, for the same initial velocity, the maximum range of flight does not exceed 3.2 km, i.e., is reduced to less than 1/20 of the theoretical value. The maximum height attained by the bullet is reduced almost in the same proportion.

The most advantageous angle of firing approaches 45°. Long-range guns fire at an angle close to 45°. Since projectiles rise in this case to a larger height, where the density of the atmosphere is lower, the effect of air resistance becomes less noticeable.

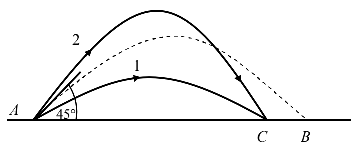


Fig. 4.45

If the target C is at a distance less than the maximum range AB (Fig. 4.45), the projectile can hit the target in two ways: at an angle of inclination which is either less than 45° (grazing firing) or larger than 45° (steep firing).

4.5. Uniform Circular Motion

Consider a point that moves along an arbitrary x- axis at velocity v_1. Subsequently the point turns at right angles and moves along y- axis at velocity v_2. The velocity of the point changed in direction and if $v_2 \neq v_1$, it changed in magnitude as well. In case $v_2 = v_1$, the change in velocity arises solely because of change in the direction of motion. What is the direction of average acceleration of the point? Figure 4.46 shows that $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ is directed inside of the curve. Hence, the average acceleration $\vec{a}_{av} = \dfrac{\Delta\vec{v}}{\Delta t}$ is also directed inside the curve.

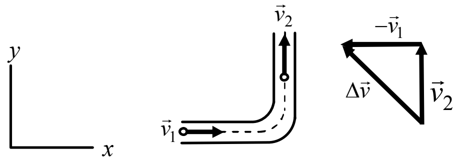


Fig. 4.46

If the point navigates a turn broken in two stages (Fig. 4.47(a)), it is subjected to two accelerations. If the turn is broken in three stages, the point will have three accelerations, as shown in (b). What about the acceleration of the point when the number of stages increases? The acceleration appears as in (c) as the number of stages is increased.

When the line segments merge into an arc of a circle, the instantaneous acceleration is directed radially inward, toward the centre. This acceleration is called the *centripetal* (centre-seeking or directed toward centre) acceleration.

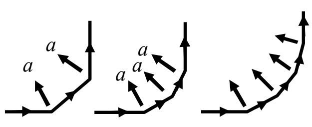


Fig. 4.47

Let us proceed to analyze the circular motion quantitatively. Figure 4.48 shows a point moving at constant speed v in a circle of radius r.

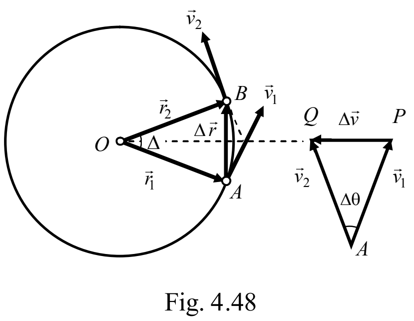


Fig. 4.48

Suppose that in a small time interval Δt the position vector of the point rotates through the angle $\Delta\theta$ and change in its position is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$. Since $\vec{v}$ is always perpendicular to $\vec{r}$, these two vectors change their directions by the same angle in any time interval. In the vector diagram for the equation $\vec{v}_2 = \vec{v}_1 + \Delta\vec{v}$, we know that $|\vec{v}_2| = |\vec{v}_1|$. The direction of $\Delta\vec{v}$ is perpendicular to $\Delta\vec{r}$ directed toward the centre O of circle, radially inward along the bisector of the angle $\Delta\theta$ drawn within the circle. The triangles OAB and APQ are similar isosceles triangles. (How?)

It follows from the similarity of the triangles that the ratios of their analogous sides are equal:

$$\frac{|\Delta\vec{r}|}{r} = \frac{|\Delta\vec{v}|}{v},$$

which gives $|\Delta\vec{v}| = \left(\dfrac{v}{r}\right)|\Delta\vec{r}|$.

Since, Δt is very small, $|\Delta\vec{r}| \approx v\Delta t$, and therefore we finally get $\dfrac{|\Delta\vec{v}|}{\Delta t} \approx \dfrac{v^2}{r}$.

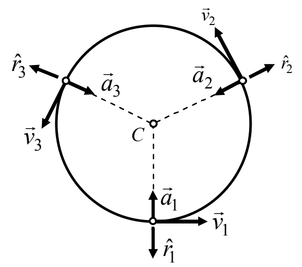
From the definition $a = \lim\limits_{\Delta t \to 0}\left(\dfrac{|\Delta\vec{v}|}{\Delta t}\right)$, we find that the magnitude of centripetal acceleration is

$$a_r = \frac{v^2}{r}. \qquad \ldots(4.17)$$

The subscript r in a_r (at times denoted by a_N also) indicates that the acceleration is in radial direction. As a vector equation we would write

$$\vec{a}_r = -\frac{v^2}{r}\hat{r}$$

where $\hat{r}$ is the radial unit vector directed outward as shown in Fig. 4.49. The figure also shows the velocity and acceleration of the point at three arbitrary points on the circular path.



$$\vec{v}_1 \neq \vec{v}_2 \neq \vec{v}_3 \neq \ldots$$

$$|\vec{v}_1| = |\vec{v}_2| = |\vec{v}_3| = \ldots = v$$

$$\vec{a}_1 \neq \vec{a}_2 \neq \vec{a}_3 \neq \ldots$$

$$|\vec{a}_1| = |\vec{a}_2| = |\vec{a}_3| = \ldots = \frac{v^2}{r}$$

Fig. 4.49

Can you write the expression for centripetal acceleration of the point in terms of time period? The *period* T is the time it takes to complete one revolution. The point travels a distance of $2\pi r$ in one revolution, so the speed is $v = (2\pi r)/T$.

As an alternative, the above derivation can be done in an absolutely simple way using calculus.

Consider a point moving on a circle with constant angular velocity $\omega = \frac{v}{r}$, where v is the speed of the point and r is the radius of the circle, Fig. 4.50.

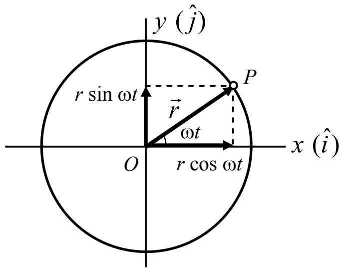


Fig. 4.50

If the point is on positive x- axis at time $t = 0$, $\angle POx = \omega t$. The position vector $\vec{r}$ of the point can be written in terms of unit vectors $\hat{i}$ and $\hat{j}$ as

$$\vec{r} = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}.$$

On differentiating the position vector $\vec{r}$ relative to time twice we get the acceleration vector. Note that r and ω are constant.

$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega r\sin\omega t\,\hat{i} + \omega r\cos\omega t\,\hat{j}$$

and $$\vec{a} = \frac{d\vec{v}}{dt} = -\omega^2 r\cos\omega t\,\hat{i} - \omega^2 r\sin\omega t\,\hat{j}.$$

By looking at the components of $\vec{a}$ we immediately conclude that its magnitude is $\omega^2 r = \left(\frac{v}{r}\right)^2 r = \frac{v^2}{r}$ and it must be directed opposite to $\vec{r}$. Vector $\vec{r}$ is directed from centre O to the point P, therefore, $\vec{a}$ must be directed from point P to the centre O. See Fig. 4.51.

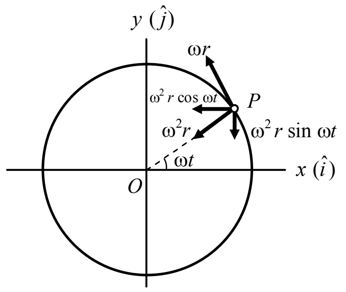


Fig. 4.51

From the expression for the velocity vector $\vec{v}$ you can conclude that the magnitude of velocity is ωr and it is directed along the tangent to the circle as shown in the figure.

Another derivation that employs polar coordinates is given in Appendix 1. Students are advised to learn these derivations, as they turn out to be a formidable tool for problem solving.

▶ **Example 12.** The coordinates of a particle at time t are given by $x = 3\sin 5t$ and $y = 3\cos 5t$. What is the speed of the particle?

To get v_x and v_y we differentiate the corresponding x- and y- coordinates

$$v_x = \frac{dx}{dt} = 15\cos 5t,\ \ v_y = \frac{dy}{dt} = -15\sin 5t.$$

Hence the speed of the particle

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{15^2\cos^2 5t + 15^2\sin^2 5t} = 15 \text{ units.}$$

Show that the trajectory of the particle is a circle. ◀

4.5.1. Acceleration in Curvilinear Motion

Consider a point moving along a curved path, as shown in Fig. 4.52. In general, both the magnitude and the direction of the velocity may vary along its path. The radial acceleration associated with changes in the direction of the velocity is $a_r = \frac{v^2}{r}$, directed toward the centre of curvature, where r is the radius of curvature of the path at the given point. A small segment of the path can be treated as an arc of a circle. The figure shows one such circle. The radius of the circle thus generated is the radius of curvature of the path at that point and its centre is the centre of curvature.

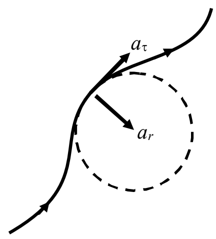


Fig. 4.52

When the speed of the point varies with time, there is also an acceleration along the tangent to the path:

$$a_\tau = \frac{dv}{dt}, \qquad \ldots(4.18)$$

in the direction of the velocity if the speed is increasing, and opposite to velocity if the speed is decreasing. The resultant acceleration of the point is the vector sum of these two components:

$$\vec{a} = \vec{a}_r + \vec{a}_\tau.$$

Since $\vec{a}_r$ and $\vec{a}_\tau$ are always perpendicular, the magnitude of the resultant acceleration is

$$a = \sqrt{a_r^2 + a_\tau^2}.$$

In the special case of motion of a point in a circle, it is sometimes convenient to use the unit vectors $\hat{\theta}$ and $\hat{r}$ shown in Fig. 4.53, where $\hat{r}$ is directed radially outward from the centre and $\hat{\theta}$ is in the direction of increasing θ. The magnitudes of these unit vectors are constant (equal to unity), but their directions change in time. The acceleration of the point is expressed as

$$\vec{a} = -\frac{v^2}{r}\hat{r} + \frac{dv}{dt}\hat{\theta}. \qquad \ldots(4.19)$$

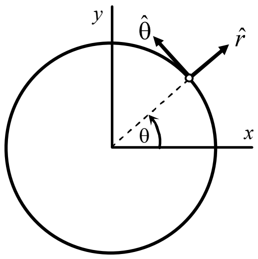


Fig. 4.53

In uniform circular motion $dv/dt = 0$, so the acceleration has only the radial term.

❓ Consider a particle moving in a circle with a variable speed, which means the angular velocity ω varies with time. From the expression $\vec{r} = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$ derive Eq. 4.19.

▶ **Example 13.** The tangential acceleration a_τ of a particle moving in a circle of radius 2 m varies with time t as in Fig. 4.54. Initial velocity of particle is zero. Find the time after which the total acceleration of the particle makes an angle of 45° with velocity?

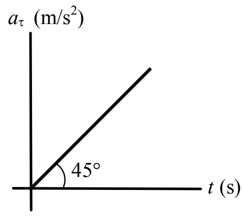


Fig. 4.54

From the graph we can express the tangential acceleration a_τ as a function of time t. We have

$$a_\tau = (\tan 45^\circ)t$$

or $\quad a_\tau = t$

or $\quad \dfrac{dv}{dt} = t.$

On integration we get

$$v = \frac{t^2}{2} + C,$$

where C is a constant of integration.

At $t = 0$, $v = 0$, hence $C = 0$.

Thus the speed of the particle as a function of time is

$$v = \frac{t^2}{2}.$$

The radial acceleration of the particle is

$$a_r = \frac{v^2}{r} = \frac{t^4}{8}.$$

The angle (which is given to be 45° at instant t) between total acceleration and velocity is same as the angle between total acceleration and tangential acceleration, see Fig. 4.55. (Is this always true? What if the particle were to decelerate?)

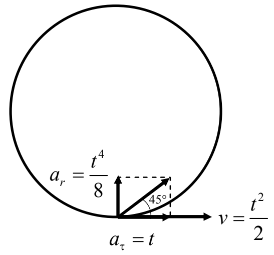


Fig. 4.55

From the figure we have

$$\tan 45^\circ = \frac{\frac{t^4}{8}}{t} = \frac{t^3}{8}$$

which gives $t = 2$ s. ◀

4.6. Relative Velocity

We shall now study the cases when one of the reference frames is in motion relative to another frame. Clearly, the second reference frame is also in motion relative to the first one.

To begin with students are advised to read the following questions, and try to answer them intuitively or otherwise, before learning the concept and formulations of the relative velocity formally. You may not be able to answer them correctly, you must try nevertheless.

(i) A girl is in a rectilinear non-uniform motion on a platform and the platform moves uniformly in a straight line relative to the ground, can the motion of the girl as seen from the ground be along a curved path?

(ii) Under what condition is the motion of the girl rectilinear in (i), even if she is in non - uniform rectilinear motion on the platform?

(iii) A boat covers a certain distance downstream in less time than it takes to cover the same distance upstream. Why?

(iv) A swimmer covers 3 km in 3 hours in still water, and a log covers 1 km downstream during the same time. What distance will be covered by the swimmer upstream in the same time?

(v) A crawler tractor moves at a velocity of 5 m/s. What is the velocity of the (a) upper, and (b) lower parts of the crawler relative to the ground? What are the velocities of these parts relative to the tractor?

The motion of any body has to be described relative to some frame of reference, such as the ground. Sometimes it is necessary to examine the motion of one body relative to another body that is also moving relative to the ground. For one-dimensional motion, it is easy to determine the velocity of one body relative to another. As an example, consider the rectilinear motion of two points.

Assume that two points A and B move on parallel straight lines with velocities 1 m/s and 2 m/s respectively relative to the ground. Let the points be side by side at the initial moment $t = 0$, Fig. 4.56.

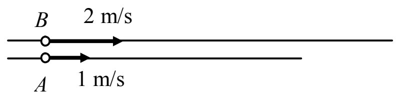


Fig. 4.56

Positions of the points A and B at time $t = 0$, 1 s, 2 s, 3 s are drawn in Fig. 4.57.

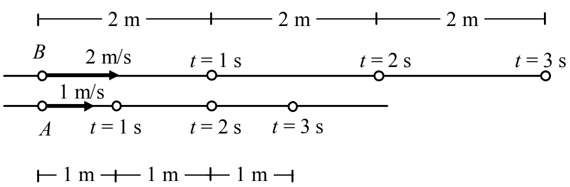


Fig. 4.57

Now, imagine an observer fixed on point A who looks at point B. Equivalently, plant yourself at A and look at B. What do you observe? At initial moment $t = 0$, point B is on your side, 1 second later B is 1 m ahead of you, 2 seconds later B is 2 m ahead of you, 3 seconds later B is 3 m ahead of you, and so on. We shall represent the position B as observed by you (which is also the position of B relative to A) by the symbol r_{BA} , and in the same notation velocity of B relative to A as v_{BA}.

The position of point B relative to point A is shown in Fig. 4.58(a) and plotted in (b).

We shall apply Eq. 4.1 to the history of motion of point B as 'seen' from point A to compute velocity of B relative to A as

$$v_{BA} = \frac{1\text{ m} - 0}{1\text{ s} - 0} = \frac{2\text{ m} - 0}{2\text{ s} - 0} = \frac{3\text{ m} - 0}{3\text{s} - 0} = 1\text{m/ s}.$$

If you subtract the velocity of A relative to the ground from the velocity of B relative to the ground, you get 1 m/s:

$$v_{BG} - v_{AG} = 2\text{ m/s} - 1\text{ m/s} = 1\text{ m/s}.$$

This is exactly the velocity of B as calculated by the observer on A from the position versus time graph.

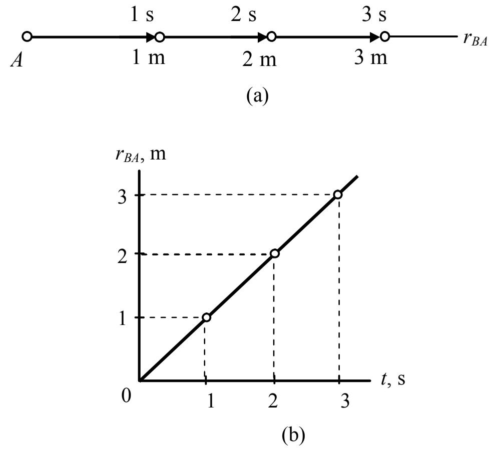


Fig. 4.58

Therefore, in this example you can obtain the velocity of point B relative to point A by subtracting the velocity of A relative to ground from the velocity of point B relative to ground.

$$v_{BA} = v_{BG} - v_{AG}.$$

Proceeding in the similar way, we can obtain the position vector of A relative B at moments $t = 0$, 1 s, 2 s, 3 s…, (Fig. 4.59(a)), and can generate the position versus time graph for the motion of point A as seen from the reference frame fixed at point B as in (b).

Computation using the graph of Fig. 4.59(b) shows that the velocity of A relative to B is –1 m/s. Also, $v_{AB} - v_{BG} = 1\text{ m/s} - 2\text{ m/s} = -1\text{ m/s}$. Hence, we conclude that

$$v_{AB} = v_{AG} - v_{BG}.$$

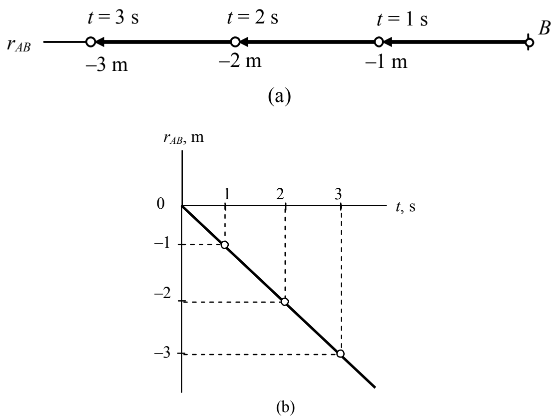
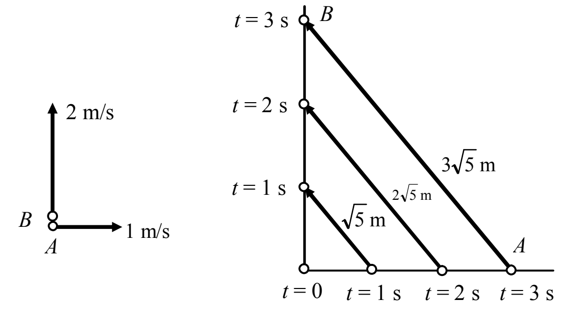


Fig. 4.59

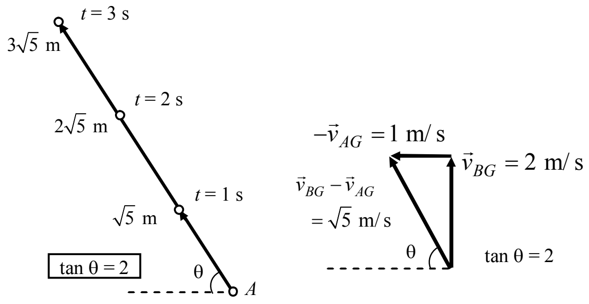
Now we shall consider the case in which the points A and B move in a plane in different directions. As a specific example, let us assume that A and B move in mutually perpendicular directions with constant velocities 1 m/s and 2 m/s as shown in Fig. 4.60(a). We also assume that the positions of A and B coincide at the initial moment $t = 0$. The positions of points A and B are drawn at the moments $t = 0$, 1 s, 2 s, 3 s in figure (b).



(a) (b)

Fig. 4.60

The position vectors of point B as seen from A at the moments $t = 0$, 1 s, 2 s, 3 s are drawn in Fig. 4.61(a).



(a) (b)

Fig. 4.61

Now, the velocity of point B relative to A can be obtained by dividing the changes in position vector by the corresponding time intervals. That is,

$$\vec{v}_{BA} = \frac{\sqrt{5}\ \text{m} - 0}{1\ \text{s} - 0} = \frac{2\sqrt{5}\ \text{m} - 0}{2\ \text{s} - 0} = \frac{3\sqrt{5}\ \text{m} - 0}{3\ \text{s} - 0} = \sqrt{5}\ \text{m/s},$$

in the direction of change in position vector, at an angle θ where $\tan\theta = 2$, as shown in the figure.

We can show that we get this same velocity by subtracting the velocity of A relative to the ground from the velocity of B relative to the ground, Fig. 4.61(b).

Arguing in the same fashion we can show that on subtracting the velocity of B relative to the ground from the velocity of A relative to the ground we get the velocity of A relative to B

$$\vec{v}_{AG} - \vec{v}_{BG} = \vec{v}_{AB}.$$

In the above examples the points were in uniform motion. Now we shall show that the above formula is valid for non-uniform motions as well if we assume that $\vec{v}_{AB}, \vec{v}_{AG}$ and $\vec{v}_{BG}$ are the instantaneous velocities taken at the same instant of time. Let the position vectors of the point A and B from G (an origin on the ground) be denoted by $\vec{r}_{AG}$ and $\vec{r}_{BG}$. See Fig. 4.62.

Let us denote the position vector of A from B by $\vec{r}_{AB}$. These position vectors are related as

$$\vec{r}_{BG} + \vec{r}_{AB} = \vec{r}_{AG}$$

or $\quad \vec{r}_{AB} = \vec{r}_{AG} - \vec{r}_{BG}$.

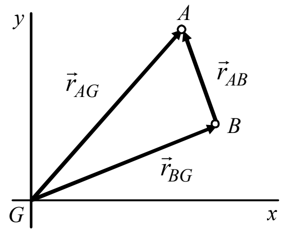


Fig. 4.62

On differentiating both sides of the above equation relative to time we get the relationship between the velocities -

$$\boldsymbol{\vec{v}_{AB} = \vec{v}_{AG} - \vec{v}_{BG}}. \qquad \ldots(4.20)$$

Let us apply the concept of relative motion to a few simple problems.

▶ **Example 14.** Let us solve Example 8 of Chapter 3 in a reference frame fixed at point A.

Calculations in the reference frame fixed at the point A:

Initial velocity of B, $u_{BA} = -4$ m/s.

Initial position of B, $r_{BA} = +6$ m.

Acceleration of B, $a_{BA} = +1.2$ m/s^2

At the moment points B hits A, r_{BA} become zero.

Hence, displacement of B is $S_{BA} = -6$ m.

Using equation $S = ut + \frac{1}{2}at^2$ we get

$$-6\ \text{m} = (-4\ \text{m/s})t + \frac{1}{2}(+1.2\ \text{m/s}^2)\ t^2$$

which gives $t = 2.3$ s, 4.4 s.

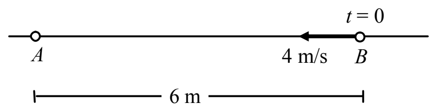


Fig. 4.63

Let us visualize the motion of point B when it is observed from point A. Initially point B was located 6 m away from the A and had a velocity of 4 m/s directed toward A (Fig. 4.63). It moves towards the A with a reducing speed as the acceleration is in the opposite direction. (Initial velocity is directed towards left and acceleration towards right.) Point B crosses point A at moment $t = 2.3$s, goes backwards comes to rest momentarily, starts moving in the forward direction, crosses A again at $t = 4s$ and then surges ahead.

Example 15. Two particles *1* and *2* have position vectors $\vec{r}_1$ and $\vec{r}_2$ at time t, where $\vec{r}_1 = (1+t^2)\hat{i} + \hat{j}$ and $\vec{r}_2 = 3\hat{i} - t^2\hat{j}$ where r is in meters and t is in seconds. Both the particles start moving when $t = 0$. At what moment of time they are closest together.

The position vector of particle *2* relative to particle *1* is

$$\vec{r}_{21} = \vec{r}_2 - \vec{r}_1 = (2-t^2)\hat{i} + (-t^2-1)\hat{j}.$$

The distance d between the particle *1* and *2* is given by

$$d^2 = (2-t^2)^2 + (-t^2-1)^2$$
$$= 2t^4 - 2t^2 + 5.$$

d^2 has stationary values when $\frac{dd^2}{dt} = 0,$

or $\quad 8t^3 - 4t = 0 \Rightarrow t = 0,\ \pm\frac{1}{\sqrt{2}}.$

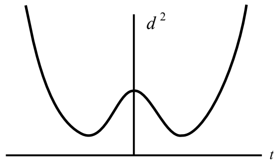


Fig. 4.64

The graph of d^2 versus t (Fig. 4.64) shows that d^2 is minimum when $t = \pm\frac{1}{\sqrt{2}}$. Therefore, the particle *1* and *2* come to closest together at $t = \frac{1}{\sqrt{2}}$ s.

? What is the least separation between the particles?

Example 16. Two stones are thrown up simultaneously from the edge of a cliff 240 m high with initial speed of 10 m/s and 40 m/s, respectively. We shall draw the graph of the time variation of position of the second stone with respect to the first. (Assume stones do not rebound after hitting the ground. We will neglect air resistance, take g = 10 m/s^2 for the sake of calculations).

One can easily conclude that the stone thrown up with velocity 10 m/s will reach ground first. This is so because as long as both the stones are in motion the other stone will always have a velocity of 30 m/s in upward direction relative to it. Further, if the stones take time, t_1 and t_2 to reach the ground, using Eq. 3.12, we have $-240 = 10t_1 - \frac{1}{2}\times 10t_1^2$, which gives $t_1 = 8$ s and $-240 = 40t_2 - \frac{1}{2}\times 10t_2^2$, which gives $t_2 = 12$ s as their times of motion. At $t = 8$ s, the first stone hits the ground and comes to rest; the second stone continues moving for another 4 seconds. This demands us to calculate the relative position of second stone from the first in two time intervals, $0 \le t \le 8$ s and $8\text{ s} < t \le 12$ s.

In time interval $0 \le t \le 8$ s, measuring the height from the point of projection,

$$y_{21} = (40t - 5t^2) - (10t - 5t^2) = 30t,$$

and in the time interval $8\text{ s} < t \le 12$ s,

$$y_{21} = (40t - 5t^2) - (-240) = 240 + 40t - 5t^2.$$

Table 4.2 depicts the essentials for generating the y_{21}-t graph, which is sketched in Fig. 4.65.

Table 4.2: Relative position of the second stone as seen from the first.

(a) $0 \le t \le 8$s					
t, s	0	2	4	6	8
y_{21}, m	0	60	120	180	240
Second stone y_{21} First stone	For $0 \le t \le 8$ s, $y_{21} = 30t$ For $8\text{s} < t \le 12$ s, $y_{21} = 240 + 40t - 5t^2$				
(b) $8\text{s} < t \le 12$s					
t, s	8	9	10	11	12
y_{21}, m	240	195	140	75	0

It is instructive to draw the variation of relative position of the second stone from the first qualitatively. An intuitive feeling of the pattern of variation of the relative position with time is as important as the step-by-step solution.

We shall now proceed to apply the concept of relative velocity to some common problems, particularly the rain-umbrella problem and the problem of crossing a river.

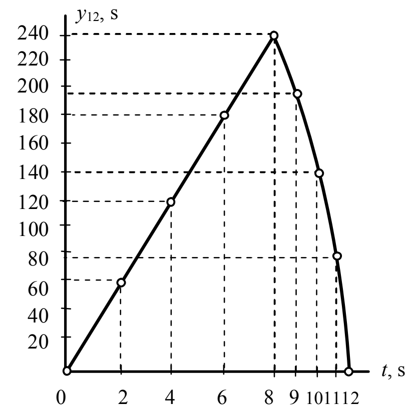


Fig. 4.65

Example 17. In the absence of winds, raindrops fall vertically downward, say with velocity $\vec{v}_{ro}$. A girl standing on the ground has to hold her umbrella in vertical position to protect herself from the rain, Fig. 4.66(a). If the girl starts walking to the right with velocity $\vec{v}_{go}$ relative to ground, in which direction she must hold her umbrella?

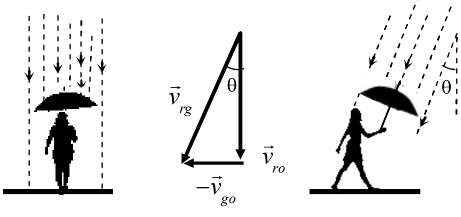
The girl will instinctively turn her umbrella and hold it at a certain angle from the vertical so as not to get drenched. We can calculate precisely the direction in which she will hold her umbrella. Formula 4.20 can be used the relate the required velocities. Velocity $\vec{v}_{rg}$ of the raindrops relative to the girl is given by the equation

$$\vec{v}_{rg} = \vec{v}_{ro} - \vec{v}_{go}.$$

As shown in Fig. 4.66(b), the raindrops appear to her falling at an angle with the vertical. We have

$$\tan\theta = \frac{v_{go}}{v_{ro}}.$$

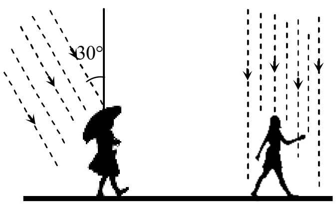
While walking, the girl will be holding her umbrella at angle θ with the vertical as shown in figure (c).



(a) (b) (c)

Fig. 4.66

Example 18. A girl standing on ground has to hold her umbrella at 30° with the vertical in order to protect herself from the rain, see Fig. 4.67(a). She throws the umbrella and starts running at 4 m/s and finds raindrops falling vertically downward (figure b). In which direction the girl is running? Find the velocity of raindrops relative to (a) the ground, (b) the running girl.



(a) (b)

Fig. 4.67

Suppose you are running on a straight road besides a cart with a speed that is equal to the speed of the cart itself. Will you surge ahead of the cart or be left behind? Answer this question and apply the same line of reasoning to this problem. The girl will find the raindrops falling vertically downward if she runs in the direction of and with the velocity of the horizontal components of the velocity of raindrops relative to the ground.

Relative to the ground the raindrops are falling at an angle of 30° with the vertical, say, with speed v_{ro}. From the velocity diagram of Fig. 4.68, you immediately get the horizontal component of the velocity of the raindrop relative to ground as $v_{ro}\sin 30^\circ$. This component must be equal to the velocity with which the girl runs. Therefore,

$$v_{ro}\sin 30^\circ = 4\,\text{m/s}$$

giving

$$v_{ro} = 8\,\text{m/s}.$$

Also, velocity of raindrops relative to girl

$$v_{rg} = v_{ro}\cos 30^\circ$$

$$= (8\,\text{m/s})\times\frac{\sqrt{3}}{2} = 4\sqrt{3}\,\text{m/s}.$$

As an effort to enhance confidence you can apply Formula 4.20 in a formal way. The velocity of raindrops with respect to ground ($\vec{v}_{ro}$), the velocity of girl relative to ground ($\vec{v}_{go}$) and velocity of raindrops with respect to girl ($\vec{v}_{rg}$) are related as

$$\vec{v}_{rg} = \vec{v}_{ro} - \vec{v}_{go}.$$

In our example, velocity of the girl relative to ground must be such that when her velocity is subtracted from $\vec{v}_{ro}$, we get a vector directed vertically downwards. Figure 4.69 illustrates the required vector operation.

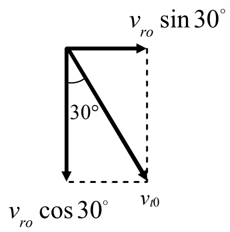


Fig. 4.68

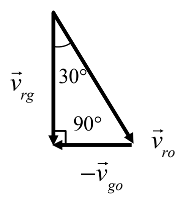


Fig. 4.69

It is left to you to calculate v_{ro} and v_{rg} from the triangle of velocities.

Example 19. A boy running in a straight line on a plain horizontal ground at 4 m/s finds that the rain drops fall at angle 30° with the vertical. He reduces his speed to 2 m/s and finds the rain falling vertically. Find the speed of the rain and the angle its velocity vector makes with the vertical.

With the assumed velocity of the rain (Fig. 4.70(a)) and given velocities of the boy (b), we construct the velocity diagram (c). That's it! This problem involves just this much physics. Now find v_{rg} and θ (in 5-6 seconds).

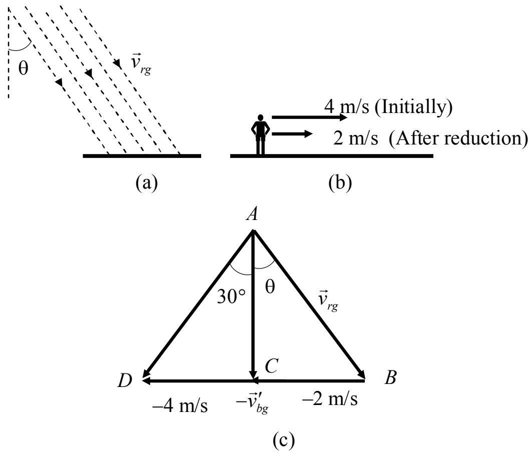


Fig. 4.70

Example 20. A boat can be sailed at speed v_1 relative to still water. The sailor wants to cross a river of width b flowing at speed v_2. (i) In which direction should the boat be headed to get straight across? How long does it take in crossing the river in this case? (ii) If the boat is pointed straight across, how long does the crossing take?

From formula 4.20, we see that the velocity of boat relative to ground $\vec{v}_{bg}$ is related to the velocity of boat relative to water $\vec{v}_1$ and the velocity of water relative to ground $\vec{v}_2$, which the river flow velocity, as

$$\vec{v}_{bg} = \vec{v}_1 + \vec{v}_2.$$

As a first step to visualize the motion of the boat consider two simple cases. If the boat is headed downstream, it will move with velocity $v_{bg} = v_1 + v_2$ in downstream direction, Fig. 4.71(a), as seen from the bank. And if the boat is headed upstream it will move with velocity $v_{bg} = v_1 - v_2$ in upstream direction as in (b).

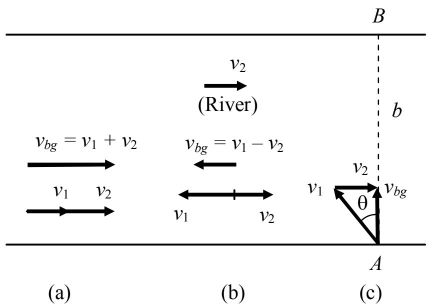


Fig. 4.71

Now, suppose the boat is at point A on one bank and the sailor wants to sail it to point B right across on the other bank taking the boat straight across. The sailor must head the boat somewhat upstream so as to compensate for the distance the boat is carried downstream by the flowing water. Indeed, the upstream component of the velocity v_1 of boat relative to water must be equal to the river flow velocity v_2. See the velocity diagram of Fig. 4.71(c). The angle θ in the figure must be such that the sum of vectors $\vec{v}_1$ and $\vec{v}_2$ is directed along line AB. What is the velocity v_{bg} with which the boat crosses the river in this case? From the velocity diagram $v_{bg} = v_1 \cos\theta$ which is equal to $\sqrt{v_1^2 - v_2^2}$ (using Pythagoras). How long does the boat take to cross the river?

$t = \dfrac{b}{\sqrt{v_1^2 - v_2^2}}$, obviously.

Can the boat cross the river in lesser time (the point where it lands on the other bank is not important)? We can use the velocity diagram (c) to answer this question. If the boat is headed as shown in the figure, it crosses the river with velocity $v_1 \cos\theta$.

Apparently, the sailor has no control over v_1 (we are analyzing the problem under the assumption that the boat is sailed on water with a constant speed) but he can choose θ as he desires. In that case, to minimize the time of crossing the river, he will try to make $v_1 \cos\theta$ as large as possible. The maximum value of the function $\cos\theta$ occurs at $\theta = 0$; therefore, the time of crossing the river is minimum when the boat is pointed straight across, see Fig. 4.72.

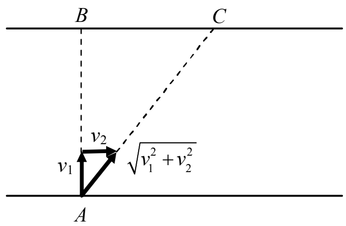


Fig. 4.72

Under this condition the boat lands on the other bank at a point somewhat downstream (at point C in the figure), and takes a time $\dfrac{AC}{\sqrt{v_1^2 + v_2^2}}$ in crossing the river. A useful result to note is that this time is also equal to $\dfrac{b}{v_1}$ or $\dfrac{BC}{v_2}$.

Example 21. A swimmer starts from point A on one bank of a river and wants to reach point B, swimming directly along line AB, see Fig. 4.73. Swimmer's speed with respect to water is 2 km/h, which is also the speed with which the river flows. Find the angle at which the swimmer must aim.

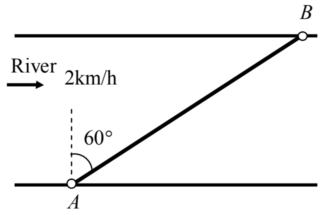


Fig. 4.73

Let the swimmer aim at angle $\theta(\theta < 60^\circ)$ as shown in Fig. 4.74. The angle θ must be such that the velocity of the swimmer relative to bank is directed along line AB.

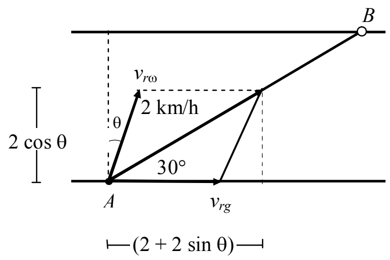


Fig. 4.74

When the velocity of river relative to ground v_{rg} is added to the velocity of the swimmer relative to water v_{sw} one obtains the velocity of the swimmer relative to ground v_{sg}. The addition of the velocities is also depicted in the figure.
From the figure we have

$$\tan 60^\circ = \frac{2+2\sin\theta}{2\cos\theta}$$

or $\quad \sqrt{3} = \dfrac{1+\sin\theta}{\cos\theta}$

or $\quad \sqrt{3}\cos\theta - \sin\theta = 1$

$\Rightarrow \quad \dfrac{\sqrt{3}}{2}\cos\theta - \dfrac{1}{2}\sin\theta = \dfrac{1}{2}$

$\Rightarrow \quad \sin 60^\circ \cos\theta - \cos 60^\circ \sin\theta = \sin 30^\circ$

or $\quad \sin(60^\circ - \theta) = \sin 30^\circ$

or $\quad 60^\circ - \theta = 30^\circ$

or $\quad \theta = 30^\circ$.

Example 22. Let us consider the problem of Example 30, Chapter 3.
We shall solve the problem in a reference frame fixed to particle *2* and moving with it.
At the initial moment $t = 0$ the separation of particle *1* from particle *2* is $\sqrt{l_1^2 + l_2^2}$ as shown in Fig. 4.75.

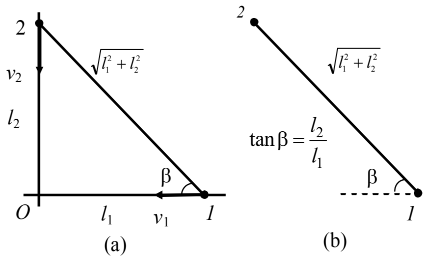


Fig. 4.75

For obtaining the velocity of particle *1* relative to the chosen reference frame subtract the velocity of particle *2* from the velocity of particle *1*, see Fig. 4.76(a).

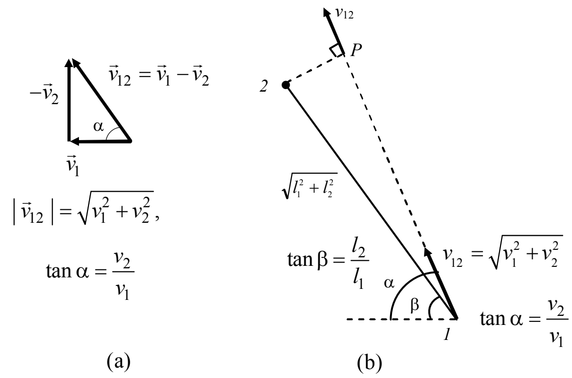


Fig. 4.76

It can be seen that particle *1* is closest to particle *2* at the instant $\vec{v}_{12}$ becomes perpendicular to the line joining them. How? In Fig. 4.76(b), the least distance between particles *1* and *2* is the length of the line 2*P*. From the figure,

$$|\tan(\alpha-\beta)| = \left|\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right| = \left|\frac{\dfrac{v_2}{v_1} - \dfrac{l_2}{l_1}}{1+\dfrac{v_2}{v_1}\cdot\dfrac{l_2}{l_1}}\right|$$

$$= \frac{|v_2 l_1 - v_1 l_2|}{v_1 l_1 + v_2 l_2}.$$

The minimum distance between the particles *1* and *2* is

$$d_{\min} = \sqrt{l_1^2 + l_2^2}\,|\sin(\alpha-\beta)|$$

$$= \sqrt{l_1^2 + l_2^2}\;\frac{|v_2 l_1 - v_1 l_2|}{\sqrt{(v_2 l_1 - v_1 l_2)^2 + (v_1 l_1 + v_2 l_2)^2}}$$

$$= \frac{|v_2 l_1 - v_1 l_2|}{\sqrt{v_1^2 + v_2^2}}.$$

We can also calculate the instant of time at which the distance between the particles is the least. To reach point P the particle *1* has to cover a distance $\sqrt{l_1^2 + l_2^2}\,|\cos(\alpha-\beta)|$ with speed $\sqrt{v_1^2 + v_2^2}$. Therefore particle *1* takes time

$$t = \frac{\sqrt{l_1^2 + l_2^2} \cdot |\cos(\alpha - \beta)|}{\sqrt{v_1^2 + v_2^2}}$$

$$= \frac{\sqrt{l_1^2 + l_2^2}}{\sqrt{v_1^2 + v_2^2}} \cdot \frac{v_1 l_1 + v_2 l_2}{\sqrt{(v_2 l_1 - v_1 l_2)^2 + (v_1 l_1 + v_2 l_2)^2}} = \frac{v_1 l_1 + v_2 l_2}{v_1^2 + v_2^2}$$

to reach point P. Since the initial moment of time is $t = 0$, the instant of time at which the distance between the two particles is the least is given by the above expression.

This approach is best suited when we are given the numerical values of the initial distances and velocities. In that case, calculations of the distance and angles is a game. As an example, take $l_1 = 10\sqrt{3}\,\text{m}$, $l_2 = 10\,\text{m}$, $v_1 = 2\,\text{m/s}$ and $v_2 = 2\sqrt{3}\,\text{m/s}$, and solve the problem. Also see Example 23.

Example 23. Two persons, *1* and *2*, are walking with constant velocities of equal magnitude along two roads which are angled at 60° to each other as shown in Fig. 4.77.

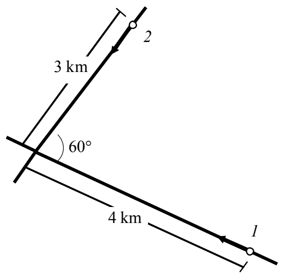


Fig. 4.77

Find the shortest distance between the persons during the course of their motion?

The problem was thrown to you by a friend standing somewhere on the ground, may be at the intersection of the roads. He expected you to use geometry to set algebra and then apply the concepts of maxima and minima of calculus and find the answer after some irritating mathematics. But you are a little smarter, not interested in doing lengthy calculations, though you can do it when asked to. All you did is this: you stepped into the shoes of person *1* and captured his sight. Whatever you saw you sketched that diagrammatically, Fig. 4.78, and came with the answer, in 10 seconds.

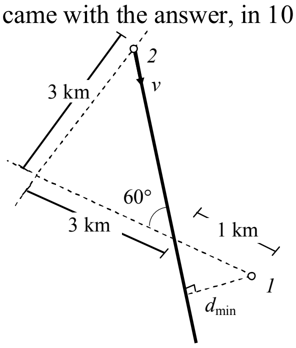


Fig. 4.78

$$d_{min} = (1\,\text{km})\sin 60^\circ = \frac{\sqrt{3}}{2}\,\text{km}.$$

Before stepping into the shoes of person *1*, you calculated the velocity of the person *2* relative to person *1*. Do it again. You determined certain angles and distance. Determine them again.

Example 24. A balloon carrying a man ascends at the rate of 3 m/s (assumed constant). At t = 3 s (assuming t = 0 to be the instant when the balloon is at the ground level), the man drops a ball (I) and simultaneously throws another ball (II) with a velocity of 1 m/s in the horizontal direction. Take $g = 10\ \text{m/s}^2$. Find the distance between the two balls at t = 4 s.

At t = 3 s, height of the balloon from the ground level = 3 m/s × 3 s = 9 m.

Velocity of the ball (I) when it is dropped = 3 m/s upwards. Ball (II) thrown with a velocity of 1 m/s in the horizontal direction also has a velocity of 3 m/s in upward direction.

Displacement of ball (I) dropped from the balloon in in one second time (from t = 3 s to t = 4 s) is

$$S = 3\ \text{m/s} \times 1\ \text{s} + \frac{1}{2}(-10\ \text{m/s}^2)\,(1\ \text{s})^2 = -2\ \text{m},$$ in vertical direction.

Horizontal component of the displacement of ball (II) in one second time is 1 m/s × 1 s = 1 m. Vertical component of the displacement of the ball (II) in one second time is –2 m. (How?) In vertical direction both the balls undergo same displacement. Therefore, the separation between the balls at t = 4 s is 1 m.

Alternatively, velocity as well as acceleration in the vertical direction of ball (II) relative to ball (I) is zero. The balls will always lie in same horizontal plane. Moreover, ball (II) does not have any acceleration relative to ball (I) in horizontal direction. Separation between the balls at $t = 4$ s is, therefore, 1 m. ◄

4.7. Velocity of approach

In Section 3.2, Chapter 3, we introduced two important quantities, namely *distance covered* by a moving point and *distance between two points*. Distance between two points or *separation between two points* is defined as the length of the straight line segment joining the points. When the distance between two points decreases, they are said to approach each other. The rate at which the points 'kill' their separation is called *velocity of approach*. To understand the concept of velocity of approach and to devise a method for its computation, we consider a few motions of two points described in the examples below.

(i) Two points A and B move along a straight line in opposite directions with speeds 3 m/s and 2 m/s, as shown in Fig. 4.79(a).

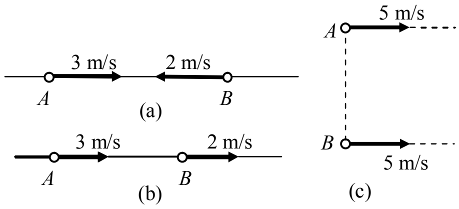


Fig. 4.79

In the given motion, points A and B come closer by 5 m in every one second time until they cross each other; the separation between them decreases by 5 m every second. We can express this by saying that the *velocity of approach* between points A and B is 5 m/s.

(ii) In Fig. 4.79(b), points A and B have velocities 3 m/s and 2 m/s in the same direction along a straight line. The distance between points A and B will decrease by 1 m every second as long as point A is behind B. Velocity of approach between points A and B is 1 m/s. If you draw their positions at instants 1 s, 2 s, 3 s…, you find that the separation between them decreases by 1 m every second.

(iii) In Fig. 4.79(c), points A and B move with equal and constant velocities, their velocities always being perpendicular to the line connecting them. Clearly, the separation between the points remains constant. The velocity of approach between them is zero, $v_a = 0$.

(iv) Consider the velocities of points A and B as shown in Fig. 4.80. Velocity of approach of points A and B is 2 m/s + 3 m/s = 5 m/s. How?

It can be shown that points A and B come closer by 5 m every second. For how long will they do it? When will they start moving apart?

You must not miss the most crucial information in this example. The velocity components of the points that are perpendicular to the line joining them are equal and in the same direction.

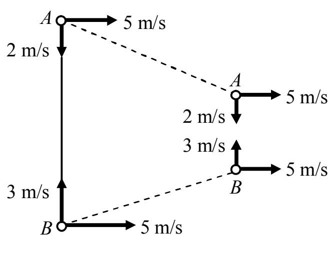


Fig. 4.80

(v) Points A and B move with a speed 5 m/s each, such that their velocity vectors are in opposite directions and always perpendicular to the line AB as shown in Fig. 4.81(a).

The length of the line segment AB will not decrease in this case. Velocity of approach between the points is zero. Both the points move in a circle, the mid point between them being the centre. It can be seen that their

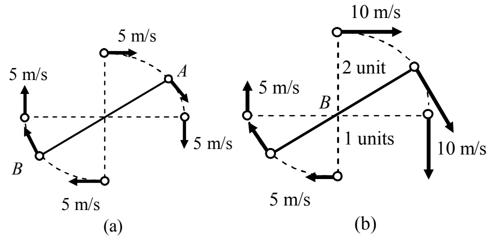


Fig. 4.81

angular velocities about the centre is same. If you plant yourself at B and look at point A, what will you observe? Point A describes a circular path? An elliptical path? Or some other complicated path? Note that the distance between points A and B does not change.

(vi) In Fig. 4.81(b), the points A and B move with speed 10 m/s and 5 m/s, respectively, their velocity vectors always being in opposite directions and also always perpendicular to the line AB. What about the separation between the two points? Will it change or not? What is the velocity of approach between the points?

(vii) Point A moves such that it has two components of velocity: one component is 5 m/s, always perpendicular to the line AB, and the other component is 2 m/s along AB.

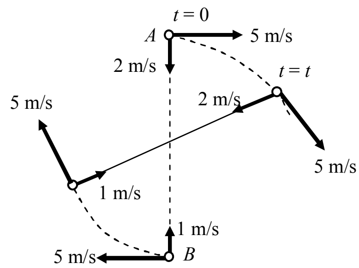
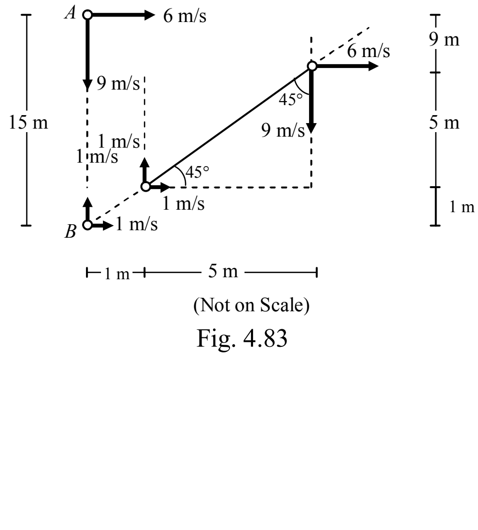


Fig. 4.82

Point B also has two components of velocity: one component is 5m/s always perpendicular to AB as shown in Fig. 4.82 and other components is 1 m/s along AB aimed at A. Velocity of approach between points A and B is 3 m/s. How?

(viii) In Fig. 4.83, the points A and B are 15 m apart initially.



(Not on Scale)

Fig. 4.83

Point A moves with constant velocity, with components 6 m/s and 9 m/s as shown. Point B also moves with a constant velocity with components 1 m/s and 1 m/s. At the initial moment (t = 0), the velocity of approach is 9 m/s + 1 m/s = 10 m/s.
What is the velocity of approach between points A and B at the moment of time t = 1 s?
To calculate the velocity of approach at t = 1 s, draw the positions of points A and B at this moment. And then resolve the velocities of A and B along the line AB and perpendicular to it. Displacements, distances and angles that matter are shown in Fig. 4.84.

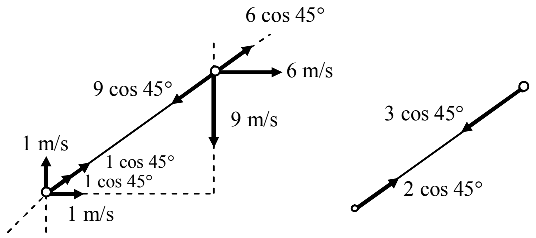


Fig. 4.84

The velocity of approach between points A and B at t = 1 s is v_a = 3 cos 45° + 2 cos 45° = 5 cos 45° m/s. At what moment of time is the velocity of approach a between A and B maximum? When do points start getting separated?

(ix)

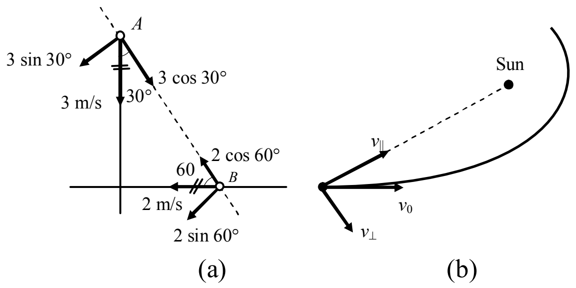


Fig. 4.85

Points A and B move in perpendicular directions as in Fig. 4.85(a). At a certain moment of time their positions and velocities are shown in the figure.
Velocity of approach between the points A and B at this moment is (3 cos 30° + 2 cos 60°) m/s.

(x) A cosmic body is approaching the Sun along a path that can be approximated to be elliptical, Fig. 4.85(b).
On looking at the cosmic body from the Sun, one finds that the length of line of sight to the body shrinks, the body comes nearer to the Sun. In the figure, the velocity of the cosmic body has been resolved into two components - the component $v_{||}$ along the line of sight, and the component $v_\perp$ perpendicular to it. The cosmic body approaches the Sun with velocity $v_{||}$. One should not have any doubt that the velocity of approach between the cosmic body and the Sun is a variable quantity in example shown in the figure. Can you predict the velocity of approach when the cosmic body is closest to the Sun? When the cosmic body is closest to the Sun, what is the angle between its radius vector from the Sun and its velocity vector?

(xi) Depicted in Fig. 4.86 is the motion of two points A and B along two parallel lines. The points come to the closest at moment t_0.

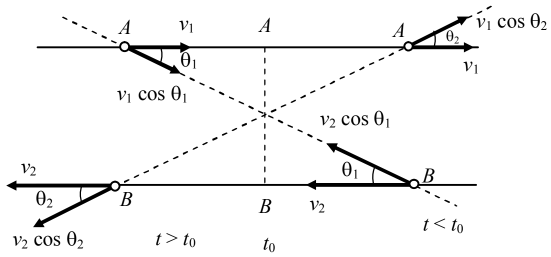


Fig. 4.86

For time $t < t_0$, the points come closer, velocity of approach between them being $v_a = v_1 \cos\theta_1 + v_2 \cos\theta_1$. How?

4.7.1. Velocity of Separation

When the distance between two points, that is, their separation increases, we speak of velocity of separation, which is defined as the rate at which the separation increases.
The examples that follow illustrate the concept of velocity of separation vividly.

(i) In the motion depicted in Fig. 4.87(a), the velocity of separation between points A and B is 5 m/s.

(ii) The velocity of separation between points A and B in the motion shown in Fig. 4.87(b) is 1 m/s.

(iii) Points A and B move with constant velocities as shown in the Figure 4.87(c). The velocity of separation is 3 m/s + 2 m/s = 5 m/s. You must note that the velocity components perpendicular to the line joining points A and B are equal.

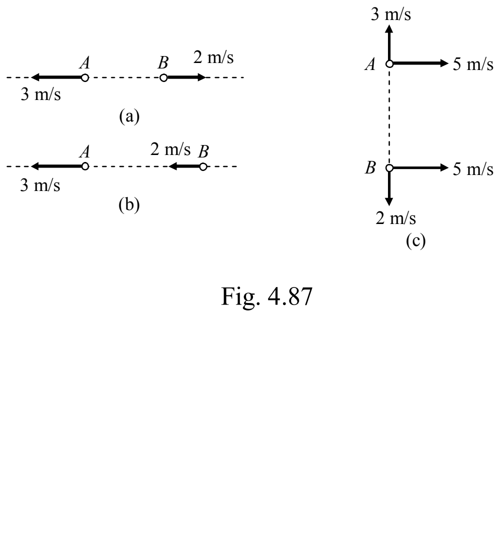


Fig. 4.87

(iv) Points A and B move with constant velocities as shown in Fig. 4.88.

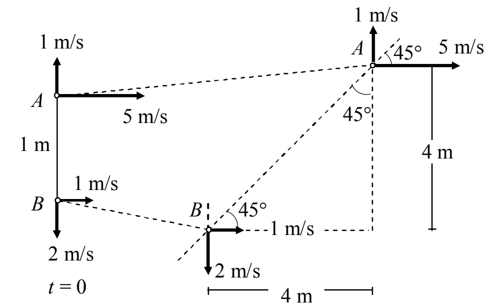


Fig. 4.88

At the initial moment $t = 0$, the points A and B are 1 m apart. What is the initial velocity of separation? It is 3 m/s. What is the velocity of separation at $t = 1$ s? Mark the positions of A and B at $t = 1$ s. Calculate the relevant distances and angles, as depicted in Fig. 4.89. Find the components of velocities along the line AB.

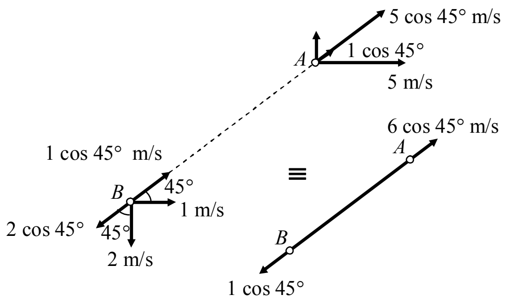


Fig. 4.89

The velocity of separation between the points A and B at time $t = 1$ s is

$$v_s = 1 \cos 45° + 6 \cos 45° = 7 \cos 45° \text{ m/s}.$$

(v) Points A and B move such that one component of velocity of A is perpendicular to the line AB and the other along AB directed away from B, and one component of velocity of B is perpendicular to AB and the other is along AB directed away from A, Fig. 4.90.

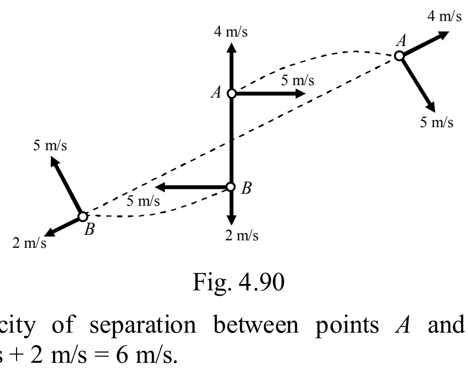


Fig. 4.90

Velocity of separation between points A and B is 4 m/s + 2 m/s = 6 m/s.

(vi) For the motion sketched in Fig. 4.86, the points move apart for time $t > t_0$. What is the velocity of separation of the points? By a look at the velocity components along the line joining the points, we arrive at the answer, $v_s = v_1 \cos\theta_2 + v_2 \cos\theta_2$.

(vii) Let us consider Example (x) of the previous section, Fig. 4.85(b). After kissing the Sun the cosmic body moves away from it. The separation between the cosmic body and the Sun now increases.

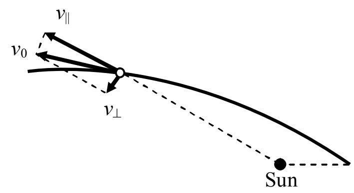


Fig. 4.91

It follows from Fig. 4.91 that the velocity of separation is $v_{\parallel}$, the component of velocity on the line connecting the cosmic body to the Sun. What will be the velocity of separation when the cosmic body is farthest from the Sun?

▶ **Example 25.** Let us take up Example 30, Chapter 3 again. We shall apply the concept of velocity of approach to this problem.

The particles 1 and 2 come closer for some time and then they go farther apart. Figure 4.92 shows the configuration at time t. The distance between them will be the least at the moment the approach velocity is zero. That is, $v_1 \cos\theta - v_2 \sin\theta = 0$

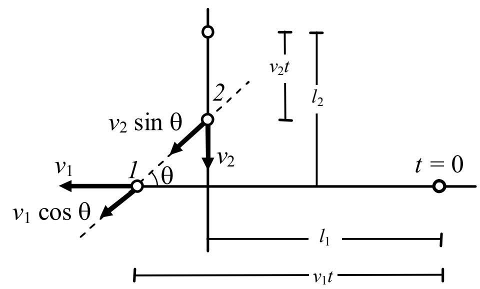


Fig. 4.92

or $$\tan\theta = \frac{v_1}{v_2}. \qquad \ldots\text{(i)}$$

Also from the figure,

$$\tan\theta = \frac{l_2 - v_2 t}{v_1 t - l_1}. \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{l_2 - v_2 t}{v_1 t - l_1} = \frac{v_1}{v_2}$$

or $$t = \frac{v_1 l_1 + v_2 l_2}{v_1^2 + v_2^2}.$$

Hence the least distance between the particles,

$$l_{\min} = \sqrt{\left(l_2 - v_2 \cdot \frac{v_1 l_1 + v_2 l_2}{v_1^2 + v_2^2}\right)^2 + \left(v_1 \cdot \frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2} - l_1\right)^2}$$

$$= \frac{|v_1 l_2 - v_2 l_1|}{\sqrt{v_1^2 + v_2^2}}.$$

Example 26. Three points are located at the vertices of an equilateral triangle whose side equals a. They all start moving simultaneously with constant speed v, with the first point heading continually for the second, the second for the third and the third for the first. How soon will the points converge?

To understand the motion of points, we treat the velocity of each point as constant for time interval dt and calculate the infinitesimal displacement. We will do it for the next dt, and then again for the next dt. When we draw the positions of the points after dt's we get the pattern of motion illustrated in Fig. 4.93.

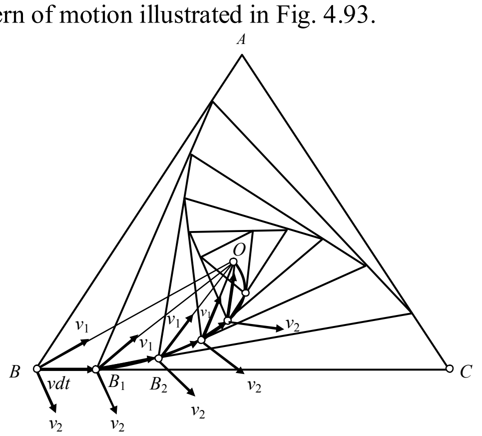


Fig. 4.93

Now let us focus on the motion of point B. If v_1 and v_2 are components of velocity of B along BO and perpendicular to it at any instant, $v_1 = v \cos 30°$ and $v_2 = v \sin 30°$. This can be inferred from the symmetry of the problem and simple geometrical calculations. Hence, we can assert that point B approaches point O at a constant rate of $v \sin 30°$. This gives

$$\text{Time of convergence} = \frac{\text{Initial separation}}{\text{Velocity of convergence}}$$

$$= \frac{\left(\dfrac{(a/2)}{\cos 30°}\right)}{v \cos 30°} = \frac{2a}{3v}.$$

We can also arrive at the answer by analyzing the convergence of points B and C, see Fig. 4.94. It can be shown that B and C converge at a constant rate of $v + v\cos 60^{\circ}$ until they meet. Hence, the time of convergence $t = \dfrac{a}{v + (v/2)} = \dfrac{2a}{3v}$.

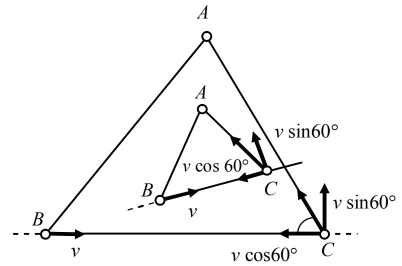


Fig. 4.94

Example 27. Point A moves uniformly with speed v so that the vector $\vec{v}$ is continually 'aimed' at point B which in its turn moves rectilinearly and uniformly with speed $u < v$. At the initial moment of time $\vec{v} \perp \vec{u}$ and points are separated by a distance l. How soon will the point converge?

Figure 4.95 sketches the motion of points A and B. If the points converge at point C, say at time $t = T$, the x-component of displacement of A must be equal to the displacement of B and the y- component of the displacement of A must be equal to l. The x and y components of velocity of A are $v_x = v \cos\theta$ and $v_y = v \sin\theta$. You can find the x- and y- components of displacement of A from the following equations.

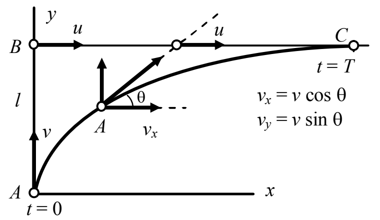


Fig. 4.95

$$x = \int_0^T v_x dt \text{ and } y = \int_0^T v_y dt.$$

Substitution of $v_x = v \cos\theta$ and $v_y = v \sin\theta$ into the integrals, and the condition of convergence of points A and B give the following two equations

$$\int_0^T v\cos\theta \; dt = uT \qquad \ldots \text{(i)}$$

and $$\int_0^T v\sin\theta \; dt = l. \qquad \ldots\text{(ii)}$$

Now what is left to be done is to solve these two equations for T.

Acknowledging that θ is a variable quantity, one can anticipate that solving for T from these equations is a formidable task. A different and easier approach to the solution is sought.

The concept of velocity of approach can be tried. Velocity of approach (or *velocity of convergence*) between points A and B is $(v - u\cos\theta)$ which is also a variable quantity. See Fig. 4.96.

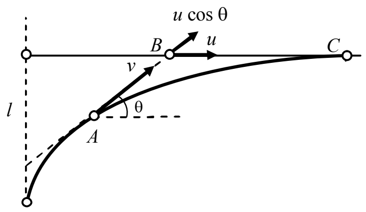


Fig. 4.96

You can calculate the distance of convergence (the separation they 'kill') in an infinitesimally small time dt by treating $v - u\cos\theta$ as constant over this interval. This infinitesimal distance of convergence equals $(v - u\cos\theta)dt$. Integration of this infinitesimal distance over total time of motion gives the total distance by which the points A and B converge, which is, obviously, equal to the initial separation l. That is,

$$l = \int_0^T (v - u\cos\theta)\ dt. \qquad \ldots\text{(iii)}$$

Eqs. (i) and (iii) can be easily solved for the unknown T. Multiply Eq. (i) by u and Eq. (iii) by v and then add the resulting equations. This gives

$$u^2T + vl = \int_0^T v^2\ dt$$

or $\quad u^2T + vl = v^2T$

or $\quad T = \dfrac{vl}{v^2 - u^2}.$

If point A moves with the same speed as that of point B, the points moving in the same fashion as described in this problem, A will not be able to catch B. Eventually both A and B will be moving on the given straight line, one point behind the other. What is the final separation between the points?

Another very interesting problem on the concept of velocity of approach is discussed in Example 37. ◀

Additional Examples

Example 28. A small steel ball A is at rest on the edge of a table of height 1 m. Another steel ball B, used as the bob of a metre-long simple pendulum, is released from rest with the pendulum suspended horizontal, and swings against A as shown in Fig. 4.97. The masses of the balls are identical and the collision is elastic. Considering the motion of A only up until the moment it first hits the ground,

(i) which ball is in motion for the longer time;

(ii) which ball covers the greater distance?

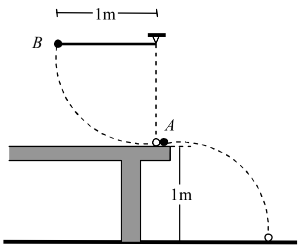


Fig. 4.97

The balls interchange their velocities, that is, ball B stops and ball A takes off with the velocity with which ball B hits it. For the time being, let us use this information and proceed. You will learn about collisions in Chapter 9 in details.

(i) Comparison of times of motion of balls:

The vertical acceleration of ball A falling from the table is always g, therefore, the time it takes to fall 1 m can be calculated as

$$1\,\text{m} = \frac{1}{2} \times 9.8\ \text{m/s}^2 \times t^2$$

which gives $t = 0.5$ s (approximately).

How long does ball B take in moving though the circular arc? The calculation of the time of motion of ball B requires an integration to be evaluated.

But it turns out that you can answer this question without actually calculating the time of motion of ball B. What can be stated with certainty is that, since the thread exerts an upward force on ball B, its vertical acceleration is always less than g. Therefore, the vertical motion of ball B takes a longer time than the vertical free fall of ball A; ball B stays in motion for longer.

(ii) Comparison of distances covered by the balls:

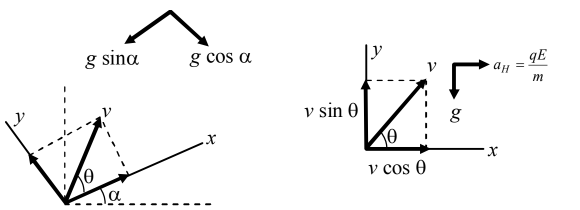
The bob of the pendulum - ball B - describes one fourth of a circle. The distance it covers is $\dfrac{\pi r}{2} = \dfrac{3.14 \times 1\,\text{m}}{2} \approx 1.5$ m. The other ball, A, follows a parabolic path, the length of which cannot be determined by elementary methods, it requires an integration. However, you can find out which ball covers a longer distance without actually calculating the distance covered by ball A. This ball hits the ground at a distance of $vt = \sqrt{2 \times g \times 1\,\text{m}} \times \sqrt{\dfrac{2 \times 1\ \text{m}}{g}} = 2$ m from the edge of the table. The length of its path is, therefore, not less than the distance between the starting point and the point where it hits the ground, which is $\sqrt{5}$ m ≈ 2.2 m.

In summary, ball A moves on a longer path, but stays in motion for a shorter time than ball B.

Example 29. A particle of mass m carries an electric charge q and is subject to the combined action of gravity and a uniform horizontal electric field of strength E. It is projected with speed v in the vertical plane parallel to the field and at an angle θ to the horizontal. What is the maximum distance the particle can travel horizontally before it is next level with its starting point?

You have investigated the motion of a projectile on an inclined plane. You have also calculated the maximum range of the projectile along the plane. This problem is very similar to inclined plane projectile problem. Figure 4.98 depicts the similarity and also the differences in motion in the two cases.

Since the particle carries a change q, in electric field of E (horizontal); there is a horizontal acceleration of $a_H = \frac{qE}{m}$ in the particle. (This is additional information which you haven't learnt yet. The concept of electric field will be introduced in Volume 3 of this text. The expression for horizontal acceleration should not frighten you.) The rest is just about doing simple kinematics under constant horizontal acceleration a_H along with vertical acceleration g due to gravity.



$v_{0x} = v\cos\theta,\ a_x = -g\sin\alpha$

$v_{0y} = v\sin\theta,\ a_y = -g\cos\alpha$

$v_{0x} = v\cos\theta,\ a_x = \frac{qE}{m}$

$v_{0y} = v\sin\theta,\ a_y = -g$

Fig. 4.98

As we do in all projectile motions, we will analyze the two components of the motion - along x- axis and along y- axis separately and independently. Let the particle hit the ground at point P such that $OP = R$, Fig. 4.99.

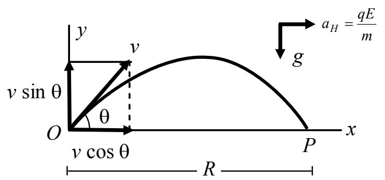


Fig. 4.99

For the x- component of motion:

$$R = (v\cos\theta)t + \frac{1}{2}a_H t^2. \qquad \ldots\text{(i)}$$

At point P the y- component of displacement is zero. Therefore, for the y- component of motion:

$$0 = v\sin\theta t - \frac{1}{2}gt^2. \qquad \ldots\text{(ii)}$$

From these equations we will calculate R in terms of θ; and then set $\frac{dR}{d\theta} = 0$ to determine θ for which R is maximum.

Eq. (ii) gives $t = \frac{2v\sin\theta}{g}$. On substituting this value of t in Eq. (i), we obtain

$$R = (v\cos\theta)\frac{2v\sin\theta}{g} + \frac{1}{2}a_H\left(\frac{4v^2\sin^2\theta}{g^2}\right)$$

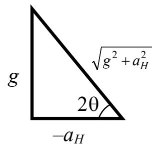
or $$R = \frac{v^2}{g}\sin 2\theta + \frac{a_H v^2}{g^2}(1-\cos 2\theta).$$

Now, $\frac{dR}{d\theta} = 0$, gives

$$\frac{2v^2}{g}\cos 2\theta + \frac{2a_H v^2}{g^2}\sin 2\theta = 0$$

or $$\tan 2\theta = -\frac{g}{a_H}. \qquad \ldots\text{(iii)}$$

For the value of θ given by Eq. (iii), the distance particle travels horizontally before it is next in level with its starting point is maximum. An interesting point must be noted here. The value of $\tan 2\theta$ is negative; this means that 2θ lies in the second quadrant, which implies $\theta > \frac{\pi}{4}$.



Now we can calculate the maximum value of the horizontal distance as

$$R_{\max} = \frac{v^2}{g}\cdot\frac{g}{\sqrt{g^2+a_H^2}} + \frac{a_H v^2}{g^2}\left(1-\left(\frac{-a_H}{\sqrt{g^2+a_H^2}}\right)\right)$$

After some simple algebra, the above equation simplifies to

$$R_{\max} = \frac{v^2}{g^2}\left(a_H + \sqrt{g^2+a_H^2}\right).$$

On substituting $a_H = \frac{qE}{m}$ into the expression for $R_{\max}$ we get

$$R_{\max} = \frac{v^2}{g^2}\left(\frac{qE}{m} + \sqrt{g^2+\left(\frac{qE}{m}\right)^2}\right)$$

$$= \frac{v^2}{mg^2}\left(qE + \sqrt{m^2g^2+q^2E^2}\right).$$

Example 30. Let us make the problem of Example 24 a bit more interesting. Assume the position of the balloon at t = 3 s as the origin of a coordinate system with x- axis in the horizontal and y- axis in the vertical direction. A wall has been constructed whose intersection with the trajectory of ball (II) is described with by equation $y = h - x\tan\alpha$, where $\alpha = 30°$, and $h = \frac{1}{2}$m as shown in Fig. 4.100. At what point (x, y) does the ball strike the wall?

Equation of the above mentioned line on the wall is $y = h - x\tan\alpha$

or $\quad y = \frac{1}{2} - \frac{x}{\sqrt{3}}$.

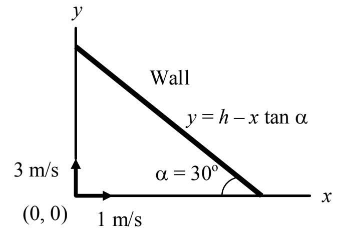


Fig. 4.100

The equations of motion of the ball (II) are

$$x = (1\,\text{m/s})t = t$$

$$y = (3\,\text{m/s})t - \frac{1}{2}(10\,\text{m/s}^2)t^2 = 3t - 5t^2.$$

Substituting for t from the first equation into the second, we obtain equation of the trajectory

$$y = 3x - 5x^2.$$

On solving the equation of the line with the equation of trajectory, we obtain

$$\frac{1}{2} - \frac{x}{\sqrt{3}} = 3x - 5x^2$$

or $\quad (5\sqrt{3})x^2 - (1 + 3\sqrt{3})x + \frac{1}{2}\sqrt{3} = 0.$

Roots of the above quadratic are

$$x = 0.23 \text{ m and } 0.56 \text{ m}.$$

Two values of x arise because the trajectory of the ball (II) is parabolic which would have intersected the line $y = h - x\tan\alpha$ at two points had this been just a line, not a line on the wall. In the given arrangement ball will hit the wall only once, so only the first impact point is to be considered for which $x = 0.23$ m, and correspondingly

$$y = \frac{1}{2} - \frac{0.23}{\sqrt{3}} = 0.37 \text{ m}.$$

Ball strikes the wall at point (0.23, 0.37) m.

Example 31. At what angle with horizontal should a particle be thrown from the origin $O(0,0)$ with a velocity v relative to the ground so that it takes minimum time to hit the line $y = h - x\tan\alpha$, see Fig. 4.101? Also, find the limiting value of v.

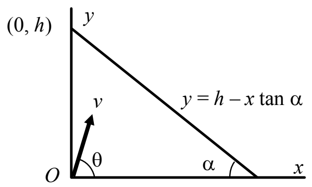


Fig. 4.101

Let the ball be projected at an angle θ with the horizontal and strike the wall in time t.

The laws of motion for the ball are

$$x = (v\cos\theta)t$$

$$y = (v\sin\theta)t - \frac{1}{2}gt^2.$$

Equation of the wall is

$$y = h - x\tan\alpha.$$

On eliminating x and y from these equations we obtain

$$\frac{1}{2}gt^2 - v[\cos\theta\tan\alpha + \sin\theta]t + h = 0$$

or $\quad \frac{1}{2}gt^2 - v\left[\frac{\cos\theta\sin\alpha + \sin\theta\cos\alpha}{\cos\alpha}\right]t + h = 0$

or $\quad \frac{1}{2}gt^2 - \frac{v}{\cos\alpha}[\sin(\theta+\alpha)]t + h = 0. \qquad \ldots\text{(i)}$

Differentiating the above equation relative to θ,

$$\frac{1}{2}g \times 2t\frac{dt}{d\theta} - \frac{v}{\cos\alpha}\left[t\cos(\theta+\alpha) + \sin(\theta+\alpha)\frac{dt}{d\theta}\right] = 0$$

For t to be minimum, $\frac{dt}{d\theta} = 0$. On substituting $\frac{dt}{d\theta} = 0$ in the above equation we obtain

$$-\frac{v}{\cos\alpha}[\cos(\theta+\alpha)]t = 0$$

or $\quad \cos(\theta+\alpha) = 0$

$\Rightarrow \quad \theta + \alpha = 90°$

or $\quad \theta = 90° - \alpha.$

Substituting $\theta = 90° - \alpha$ into Eq. (i) and solving for t we obtain

$$t_{\text{min}} = \frac{v - \sqrt{v^2 - 2gh\cos^2\alpha}}{g\cos\alpha}.$$

It can be seen that t_{min} will have a real, meaningful value only if

$$v^2 > 2gh\cos^2\alpha$$

or $\quad v > \sqrt{2gh}\cos\alpha.$

Example 32. On a frictionless horizontal surface, assumed to be the *x-y* plane, a particle A is moving along a straight line parallel to y- axis with a constant velocity of $(\sqrt{3} - 1)$ m/s. At a particular instant when the line OA makes an angle of 45° with the x- axis, another particle B is projected along the surface from the origin O, see Fig. 4.102. Its velocity makes an angle ϕ with the x- axis and it hits particle A. (a) How must the velocity v of particle B be related to the angle ϕ for this to happen? Can you determine the values of ϕ and v uniquely on the basis of forgoing given information? **(Continued….)**