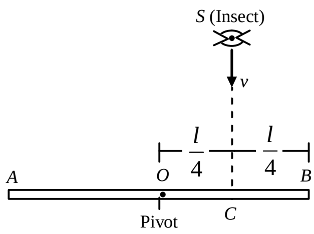
Rigorous Analysis…

SP Neelam, B Tech (IT BHU), M Tech (IIT Roorkee)

1. A homogeneous rod *AB* of length $l = 1.8$ m and mass *m* is pivoted at the centre *O* in such a way that it can rotate freely in the vertical plane, see the figure. The rod is initially in the horizontal position. An insect *S* of the same mass *m* falls vertically with speed *v* on the point *C*, mid way between the points *O* and *B*.
Immediately after falling, the insect moves towards the end *B* such that the rod rotates with a constant angular velocity ω.
(a) Determine the angular velocity ω in terms of *v* and *l*.
(b) If the insect reaches the end *B* when the rod has turned through an angle of 90^o, determine *v*.
(Take $g = 10$ m/s^2.) (IIT-JEE 1992, 8 Marks)

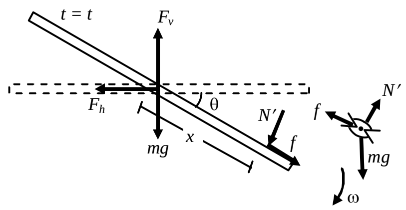


Solution : The mass of the *insect* is equal to the mass of a 1.8-m long *rod*. A very large insect indeed! Large insects have masses 50-70 grams. But their lengths vary from 10 to 15 centimeters. Here, the insect has to be treated as a particle. The clauses '…falls vertically with speed *v* on the point *C*' and '…reaches the end *B*' will assume meaning only in that case.

Also, the motion of the 'rod-insect' system, or only rod or only insect for that matter, must be governed by the laws of Particle Dynamics. The forces on the system, or on its parts, are the usual forces we study - the weight, the normal force, the frictional force, etc. However, a rigorous analysis shows that unknown forces are present here. The insect makes special efforts. The motion of the insect is in no way related to the motion of the rod except that the insect somehow remains in contact with the rod and moves towards its end *B*.

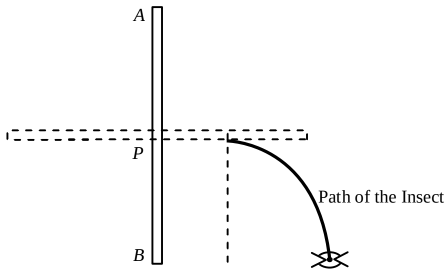
First let us apply the laws of Particle Dynamics.

After falling on the rod the insect moves towards the end *B*, *such that* the rod rotates with a constant angular velocity. This implies that the angular acceleration of the rod about the pivot is zero. And this will happen if the torque on the rod about the pivot is zero.
Consider the forces on the rod when it rotates about its pivot.



For the torque on the rod about the pivot *O* to be zero, the normal force N' on the rod by the insect must be equal to zero. And in that case, generally speaking, the frictional force f will also be equal to zero.

Now look at the forces acting on the insect. The normal force $N' = 0$, the frictional force $f = 0$. So the only force of the insect is its weight mg. The insect starts moving towards the end B of the rod immediately after falling on it. Under these conditions the path followed by the insect must be somewhat as shown in the figure.



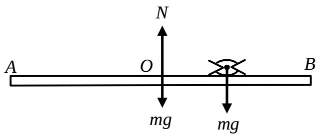
If this is so, how can the insect reach end B when the rod has turned through 90° angle? So if the rod were to rotate with a constant angular velocity the insect would be moving as a projectile under gravity, taking the parabolic path as shown in the figure!
But there can be a frictional force on the insect, and its reaction on the rod, without the normal force N' shown in the figure above. The insect may press the rod on its sides, and thereby generate the required frictional force. This is certainly possible. But then the insect must somehow balance the component of its weight normal to the rod. There must not be any normal force on the rod by the insect.
Also, for the insect, the force equation in the radial direction is $f - mg\sin\theta = m(\omega^2 x - a_{\text{rel}})$.

Here, a_{rel} is the acceleration of the insect relative to the rod. From this equation it can be seen that the force f varies in a very complicated way during the motion of the system.

Hence, the insect makes special efforts in generating unknown forces that cannot be accounted for.

Let us ignore all these details, treat the insect as a particle, ignore the torque of the unknown force about the pivot that the insect somehow generates if any, and solve the problem as given below.

The forces acting on the 'rod-insect' system during the impact is shown in the figure. The normal force on the rod by the pivot during the impact is an impulsive force, $N >> 2\ mg$. What all information do you need to calculate the value of the normal force N during the impact?



Here $\Sigma \vec{F}_{ext} \neq 0$, (convince yourself), so linear momentum of the 'rod-insect' system changes.

From equation $\vec{\tau} = d\vec{L}/dt$, you get $\vec{L}$ = constant if $\vec{\tau} = 0$. The normal force N on the rod by the pivot is a force not known to you. Can you locate a point about which the torque acting on the system is zero unless you know all the forces acting on the system?

Also, clearly, $\tau_{pivot} = mg.\frac{l}{4} \neq 0$. Here $\left(mg \cdot \frac{l}{4}\right)\Delta t = 0$ as Δt is negligibly small. So under 'impulse approximation' the angular momentum of the 'rod-insect' system about the pivot can be conserved. *(Why ?)*

or $\quad mv\frac{l}{4} = \left(\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right)\omega$

or $\quad mv\frac{l}{4} = \left(\frac{1}{12} + \frac{1}{16}\right)ml^2\omega$

or $\quad \omega = \frac{12}{7}\frac{v}{l}.$

The insect moves towards end B such that the rod rotates with constant angular velocity. There will always be a torque on the system about the pivot. This torque does not result in an angular acceleration otherwise ω will change. The equation $\vec{\tau} = I\vec{\alpha}$ holds for a system where the moment of inertia I about the axis of rotation is constant. Here the moment of inertia of the 'rod-insect' system changes with time.

But even if the moment of inertia about the axis of rotation changes, we can apply the equation $\vec{\tau} = \frac{d\vec{L}}{dt}$.

Using this equation we get

$$mgx\cos\theta = \frac{d}{dt}\left\{\left(\frac{ml^2}{12} + mx^2\right)\omega\right\}$$

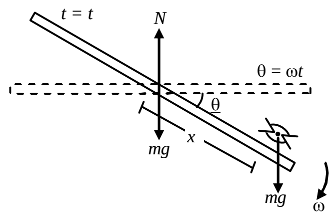
or $\quad mgx\cos\theta = 2mx\frac{dx}{dt}\omega$

or $\quad g\cos\theta = 2\omega\frac{dx}{dt}. \qquad \ldots(1)$

This is the differential equation of motion of the system. Substituting $\theta = \omega t$ in this equation,

$$g\cos\omega t = 2\omega\frac{dx}{dt}$$

or $\quad g\cos\omega t\ dt = 2\omega dx. \qquad \ldots(2)$



The rod turns through $\pi/2$ in time $(\pi/2)/\omega$ and the position of the insect changes from $x = l/4$ to $x = l/2$.

Integrating both the sides of equation (2) with these limits

$$\int_0^{\pi/2\omega} g\cos\omega t\ dt = \int_{l/4}^{l/2} 2\omega\ dx$$

or $\quad \frac{g}{\omega}|\sin\omega t|_0^{\pi/2\omega} = 2\omega|x|_{l/4}^{l/2}$

or $\quad \frac{g}{\omega}\left(\sin\omega.\frac{\pi}{2\omega} - \sin 0\right) = 2\omega\left(\frac{\ell}{2} - \frac{\ell}{4}\right)$

or $\quad \frac{g}{\omega} = 2\omega\frac{l}{4}$

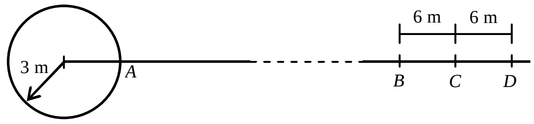
or $\quad \omega^2 = \frac{2g}{l}$

or $\quad \left(\frac{12}{7}\frac{v}{l}\right)^2 = \frac{2g}{l} \qquad \left(\omega = \frac{12}{7}\frac{v}{l}\right)$

or $\quad v = \frac{7}{12}\sqrt{2gl} = \frac{7}{12}\sqrt{2\times 10\times 1.8} = 3.5$ m/s.

2. A source of sound is moving along a circular orbit of radius 3 m with an angular velocity of 10 rad/s. A sound detector located far away from the source is executing simple harmonic motion along the line BD (see the figure) with an amplitude $BC = CD = 6$ meters. The frequency of oscillation of the detector is $5/\pi$ per second. The source is at point A when the detector is at point B. If the source emits a continuous sound wave of frequency 340 Hz find the maximum and the minimum frequencies recorded by the detector. Take the velocity of sound in air as 330 m/s.

(IIT-JEE 1990, 7 Marks)

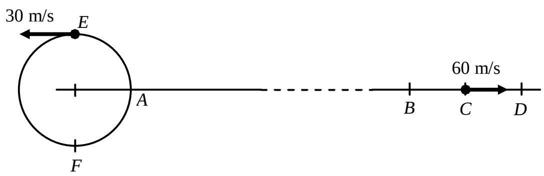


Solution: The answers, wherever the problem has been solved or given, are υ_{max} = 442 Hz and υ_{min} = 225 Hz. These answers are not correct. The logic on the basis of which they arrive at these answers has flaws. In fact, this problem cannot be solved because a crucial fact has been neglected while framing the problem.

First, let us see how these answers have been calculated.

For source	**For observer**
ω = 10 rad/s	$v = 5/\pi$ per second.
Radius of the circular orbit r = 3 m.	Angular frequency ω = $2\pi\upsilon$ = 10 rad/s.
Speed $v = r\omega$ = 30 m/s.	Amplitude of SHM = 6m
Time period $T = \frac{2\pi}{\omega} = \frac{\pi}{5}$ s.	Maximum speed, at the mean position $a\omega$ = 60 m/s.

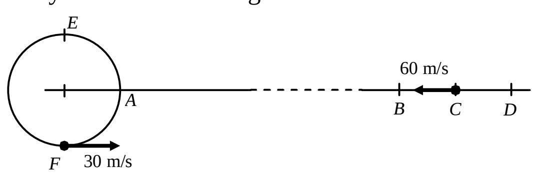
Let at time t = 0, there is no loss of generality in assuming so, the source be at point A.



The source will be at E (with velocity 30 m/s towards left) at the moments

$$t = \frac{\pi}{20}, \frac{\pi}{20}+\frac{\pi}{5}, \frac{\pi}{20}+\frac{2\pi}{5}, \frac{\pi}{20}+\frac{3\pi}{5}, \frac{\pi}{20}+\frac{4\pi}{5}, \ldots$$

And at these moments the observer will be at C with velocity 60 m/s towards right.

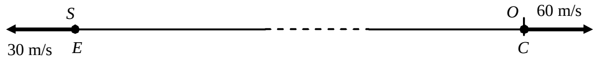


The source will be at F (velocity 30 m/s towards right) at the moments

$$t = \frac{3\pi}{20}, \frac{3\pi}{20}+\frac{\pi}{5}, \frac{3\pi}{20}+\frac{2\pi}{5}, \frac{3\pi}{20}+\frac{3\pi}{5}, \ldots$$

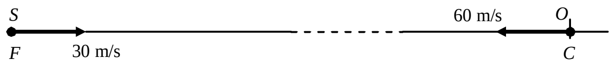
And at these moments the observer will be at C with velocity 60 m/s towards left.

As the source is far away from the observer, the velocities of the source and the observer can be taken along the same line, and the maximum and the minimum frequencies recorded can be calculated as under : For the moments when the source is at E,



$$\upsilon_{min} = \upsilon_0 \frac{v - v_o}{v + v_s} = 340 \times \frac{330-60}{330+30} = 225 \text{ Hz.}$$

And when the source is at F,



$$\upsilon_{max} = \upsilon_0 \frac{v + v_o}{v - v_s} = 340 \frac{330+60}{330-30} = 442 \text{ Hz.}$$

Now, we will check whether the given information also meets the requirements of an essential condition.

Will the note emitted by the source while at point E (moving towards left with 30 m/s) necessarily be received by the observer when it is at C (moving towards right with velocity 60 m/s)? After all the distance between the source and the observer is long enough… 'the detector located far away from the source'…; the wavelengths emitted by source when it is at point E will reach the observer only after some time, it will never reach the detector instantly. The time taken by the sound to travel from the source to the detector can not be neglected. And this time need not be an integral multiple of π/5 s!

The notes emitted by the source when it is at point E $\left(\text{at } t = \frac{\pi}{20}, \frac{\pi}{20}+\frac{\pi}{5}, \frac{\pi}{20}+\frac{2\pi}{5}, \ldots\right)$ will certainly be detected by the observer, but the position and velocity of the detector at the moment it detects these notes can not be ascertained; detector can be anywhere on the line BD, and its velocity can have any value from 60 m/s towards left to 60 m/s towards right (-60 m/s $\le v_0 \le$ +60 m/s).

Where did the examiner falter?

He combined two problems in one.

(1) A source moves along a small circle and a detector located far away on the line through the centre of the circle and in the plane of the circle. (2) A detector performs simple harmonic motion on a line and a source located far away on that line. These two problems can easily be solved. The moment of emission of a note and the moment of detection are not important here. But in the given problem they are. And the examiner did not leave the choice of selecting these moments of time to students. He clearly stated that **the source was at point *A* when the detector was at point B. Probably that is where he faltered.**

3. A cylindrical block of length 0.4 m and area of cross-section 0.04 m^2 is placed coaxially on a thin metal disc of mass 0.4 kg and of same cross-section. The upper face of the cylinder is maintained at a constant temperature of 400 K and the initial temperature of the disc is 300 K. If the thermal conductivity of the material of the cylinder is 10 Watt/m-K and the specific heat of the material of the disc is 600 J/kg-K, how long will it take for the temperature of the disc to increase to 350 K? Assume, for the purposes of calculation, the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper face of the cylinder. (IIT-JEE 1992, 8 Marks)

Solution: Here, apart from numerical values, you have been given two conditions.

(i) The thermal conductivity of the disc is very high.

(ii) The system is thermally insulated except for the upper face of the cylinder.

From the first condition you infer that the heating of the disc is uniform, there is no temperature gradient across the disc; the use of the phrase 'the temperature of the disc' is justified. The rate at which heat enters the disc, is the rate of the temperature rise of the disc given by $\frac{dq}{dt} = m\,c\,\frac{dT}{dt}$, where m and c are the mass of the disc and the specific heat of the disc material respectively.

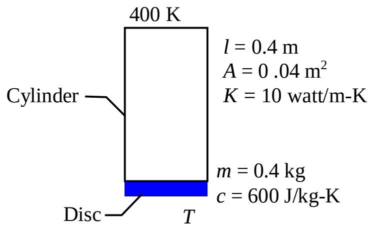
From the second condition you infer that there is no heat loss to the surrounding. All the heat that enters the top of the cylinder goes to the disc from its bottom.

But how can you account for the heat absorbed by the cylinder? After all, as the heat flow takes place, and as the temperature of the disc increases, the temperature at different cross-sections of the cylinder will also increase. Don't you think some heat would be absorbed in raising the temperature at different cross-sections of the cylinder? Of course, it would be. You can use the heat flow equation $\frac{dq}{dt} = \frac{kA(\Delta T)}{l}$ if at all the moments of time the heat flow is under steady state condition, the heat entering the upper face of the cylinder is equal to the heat leaving its bottom face, no heat being absorbed by the cylinder. Then how does the temperature at the different cross-sections of the cylinder rise?

So with the given conditions the problem can not be solved.

You can solve the problem only if you assume that the **specific heat of the material of cylinder is zero** (or negligible). That is, the temperature changes at different cross-sections of the cylinder do not involve any absorption of heat. With this condition, the problem can be solved as follows.

Let the temperature of the disc at time t be T. Let the increment of the temperature of the disc in time dt be dT.



Rate of heat flow through the cylinder = Rate of heat flow to the disc.

or $$\frac{KA(400-T)}{l} = mc\frac{dT}{dt}$$

or $$\frac{KA}{l}dt = mc\frac{dT}{400-T}$$

Integrating both sides with suitable limits,

or $$\frac{KA}{l}t = -mc \left| \ln(400-T) \right|_{300}^{350}$$

$$= -mc\,(\ln 50 - \ln 100)$$

or $$\frac{KA}{l}t = mc\ln 2$$

or $$t = \frac{lmc}{KA}\ln 2.$$

Substituting the given numerical values,

$$t = \frac{0.4\times 0.4\times 600}{10\times 0.04}\ln 2 = 240 \ln 2 \text{ s} = 166.3 \text{ s}.$$

4. An electron and a proton are moving on straight parallel paths with same velocity. They enter a semi-infinite region of uniform magnetic field perpendicular to the velocity. Which of the following statement(s) is/are true?

(A) They will never come out of the magnetic field region.

(B) They will come out traveling along parallel paths.

(C) They will come out at the same time.

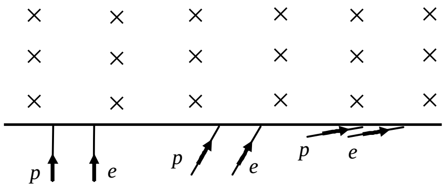
(D) They will come out at different times.

(IIT-JEE 2011, 2 Marks)

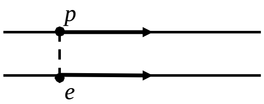
Solution: In the key given on the official IIT-JEE website the answer was **(B, C)** or **(B, D)** or **(B, C, D)**. This question has three answers! Strange isn't it? How on the earth choices (C) and (D) can be true simultaneously?

This question is very confusing. It can be easily shown that choice (B) is correct. Choice (C) may be correct, may not be correct. Similarly choice (D) may be correct, may not be correct. On the basis of the information given in the question one cannot decide it.

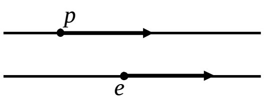
The particles enter a semi-infinite region of uniform magnetic field perpendicular to the velocity. We are given the angle the velocity vector makes with the magnetic field, but not with the boundary of the region. The following figure shows three cases out of infinite ways the particles can enter the magnetic field.



Note that the phrase "an electron and proton moving on straight parallel paths with same velocity." does not necessarily mean that the two particles are side by side as shown in the figure below.



The two particles can be moving with same velocity on parallel paths as shown in the figure below, and enter the magnetic field at the same instant.



We know that a proton is 1836 times (approximately) heavier than an electron. Once they enter the magnetic field, their angular velocities will be different due to the difference in their masses.

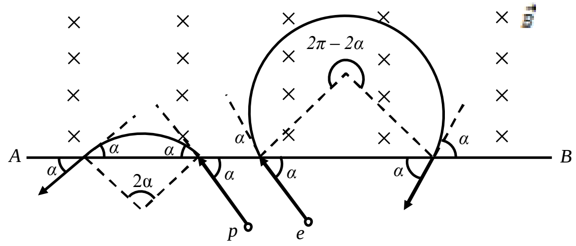
Angular velocity of proton $\omega_p = \dfrac{eB}{m_p}$, and

angular velocity of electron $\omega_e = \dfrac{eB}{m_e}$,

where B is the magnetic field, e is the magnitude of the charge on proton and electron, m_e mass of electron and m_p mass of proton.

Clearly, $\omega_e >> \omega_p$.

From this information can we really conclude that the time taken by both the particles to come out of the magnetic field will be different? Suppose the path taken by electron is longer than that taken by proton. Then there is a possibility that they come out at the same time. Consider the following situation.



Let the proton and electron enter in parallel paths at an angle α as in the figure above. AB is the boundary separating the magnetic field region from non-magnetic field region. The line AB extends to infinity.

The path of proton subtends angle 2α at the centre of the circle, while the path of electron subtends (2π -2α) at the centre of the circle. Do the required geometrical calculations yourself.

Hence time taken by the electron is

$t_e = \dfrac{(2\pi - 2\alpha)m_e}{eB}$, and that by the proton is

$t_p = \dfrac{2\alpha m_p}{eB}$.

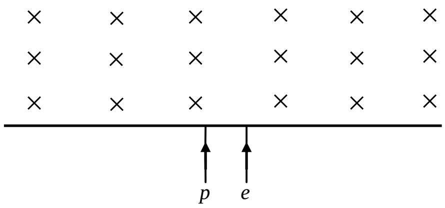
Now if $t_e = t_p$, then

$(\pi - \alpha)\, m_e = \alpha\, m_p$

or $\qquad \alpha = \dfrac{\pi m_e}{m_e + m_p} = \dfrac{\pi}{1837}$.

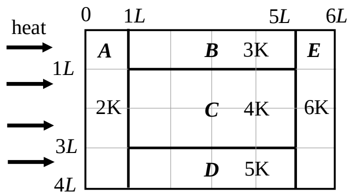
This is a very small angle. But conceptually this is a possibility. And if the particles enter the magnetic field at this angle then the time taken by them to come out of it will be equal. In that case choice (C) will be correct. In other situations choice (D) will be correct.

It seems that the examiner assumed that the velocities of electron and proton were same, they were perpendicular to the magnetic field and also **perpendicular to the boundary**, as shown in the figure below.



But it was not written in the paper clearly. Paper did not have any diagram also.

5. A composite block is made of slabs A, B, C, D and E of different thermal conductivities (given in terms of a constant K) and sizes (given in terms of length, L) as shown in the figure. All slabs are of same width. Heat 'Q' flows only from left to right through the blocks. Then in steady state



(A) heat flow through A and E slabs are same.

(B) heat flow through slab E is maximum.

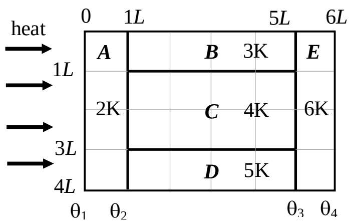
(C) temperature difference across slab E is smallest.

(D) heat flow through C = heat flow through B + heat flow through D. (IIT-JEE 2011, 3 Marks)

Solution: As per the official IIT-JEE site the answer of this question is (**A**), (**C**), (**D**). But a rigorous analysis shows that the choice (**C**) is not correct, because under the given conditions the behavior of slab E is pretty strange, and the phrase 'temperature difference across slab E ' does not have any meaning. It also turns out that the choice (**D**) is also incorrect.

This question has many flaws. If you look at the arrangement carefully, you can argue intuitively that the behaviors of slabs A and E are strange. Thermal conductivity of slab D is more than that of slabs C and B. Shouldn't heat rush to take the path of least thermal resistance? Shouldn't heat flow through slab A from top to bottom, and through E from bottom to top too? To visualize this, consider an unusual situation. Assume that the thermal conductivity of the slabs B and C is zero. Heat flow occurs only through slab D. What will be the pattern of heat flow through slabs A and E in that case?

Anyway, let us go by what has been given in the question. All the slabs allow heat flow only from left to right. For lateral heat flow they offer infinite thermal resistance. Assume the temperature at the interfaces as shown in the figure below. If the thickness of each slab is b, in steady state heat flow condition

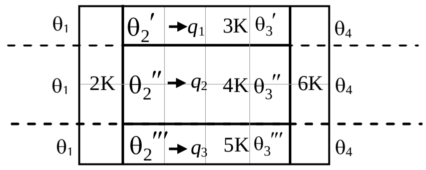


$$\frac{(2K)(4Lb)(\theta_1-\theta_2)}{L}=\frac{(3K)(Lb)(\theta_2-\theta_3)}{4L}$$
$$+\frac{(4K)(2Lb)(\theta_2-\theta_3)}{4L}+\frac{(5K)(Lb)(\theta_2-\theta_3)}{4L}$$
$$=\frac{(6K)(4Lb)(\theta_3-\theta_4)}{L}$$

$\Rightarrow 2(\theta_1-\theta_2)=(\theta_2-\theta_3)=6(\theta_3-\theta_4)$. …(1)

If θ_1 and θ_4 are fixed, you can find θ_2 and θ_3 in terms of θ_1 and θ_4.

Now since heat flows through the slabs only from left to right, you can also write heat flow equations as



$$q_1=\frac{(2K)(Lb)(\theta_1-\theta_2')}{L}=\frac{(3K)(Lb)(\theta_2'-\theta_3')}{4L}$$
$$=\frac{(6K)(Lb)(\theta_3'-\theta_4)}{L}$$

$\Rightarrow 2(\theta_1-\theta_2')=\frac{3}{4}(\theta_2'-\theta_3')=6(\theta_3'-\theta_4)$. …(2)

$$q_2=\frac{(2K)(2Lb)(\theta_1-\theta_2'')}{L}=\frac{(4K)(2Lb)(\theta_2''-\theta_3'')}{4L}$$
$$=\frac{(6K)(2Lb)(\theta_3''-\theta_4)}{L}$$

$\Rightarrow 4(\theta_1-\theta_2'')=2(\theta_2''-\theta_3'')=12(\theta_3''-\theta_4)$. …(3)

And

$$q_3=\frac{(2K)(Lb)(\theta_1-\theta_2''')}{L}=\frac{(5K)(Lb)(\theta_2'''-\theta_3''')}{4L}$$
$$=\frac{(6K)(Lb)(\theta_3'''-\theta_4)}{L}$$

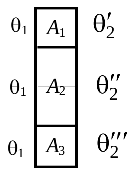
$\Rightarrow 2(\theta_1-\theta_2''')=\frac{5}{4}(\theta_2'''-\theta_3''')=6(\theta_3'''-\theta_4)$. …(4)

Look at the equations (1) through (4) carefully. Without solving them you can conclude that

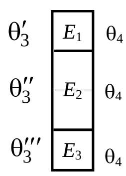
$\theta_2\neq\theta_2'\neq\theta_2''\neq\theta_2'''$ and $\theta_3\neq\theta_3'\neq\theta_3''\neq\theta_3'''$.

We can draw the following conclusions

1. Slab A is made of 3 slabs.



2. Slab E is made of 3 slabs.



3. Temperature difference across slab E has no meaning. It is $(\theta_3' - \theta_4)$ for some part, $(\theta_3'' - \theta_4)$ for some other part and $(\theta_3''' - \theta_4)$ for still other part.

So choice (C) cannot have any meaning.

4. The temperature is not same everywhere at the right end of slab A. Also the temperature is not same everywhere at the left end of slab E. Under these conditions, the formulation of equation (1) is not correct.

5. Heat flow through slabs $A_1 \to B \to E_1$

$$R_1 = \frac{L}{(2K)(Lb)} + \frac{4L}{(3K)(Lb)} + \frac{L}{(6K)(Lb)} = \frac{2}{Kb}$$

$$q_1 = \frac{\theta_1 - \theta_4}{\left(\frac{2}{Kb}\right)}.$$

Heat flow through slabs $A_2 \to C \to E_2$

$$R_2 = \frac{L}{(2K)(2Lb)} + \frac{4L}{(4K)(2Lb)} + \frac{L}{(6K)(2Lb)} = \frac{10}{12Kb}$$

$$q_2 = \frac{\theta_1 - \theta_4}{\left(\frac{10}{12Kb}\right)}.$$

Heat flow through slabs $A_3 \to D \to E_3$

$$R_3 = \frac{L}{(2K)(Lb)} + \frac{4L}{(5K)(Lb)} + \frac{L}{(6K)(Lb)} = \frac{22}{15Kb}$$

$$q_3 = \frac{\theta_1 - \theta_4}{\left(\frac{22}{15Kb}\right)}.$$

Clearly, $q_1 + q_3 \neq q_2$.

Hence choice (D) is not correct.

In the steady state, the rate of heat flow through any cross section is same $\Rightarrow$ choice (A) is correct.

We have, rate of heat flow

$$\frac{dQ}{dt} = \frac{KA(\theta_2 - \theta_1)}{l}, \qquad \ldots(1)$$

where K is thermal conductivity of the material, A is area of cross section of slab and l the length, $\theta_2 - \theta_1$ is temperature difference across the ends of slab.

Now for the slabs B, C and D the temperature difference is same.

Hence the rates of heat flow through these slabs depend on the factor $\frac{KA}{l}$.

Let us represent this factor by β.

For slab B, $\beta \equiv \frac{3K(L \times b)}{4L} = \frac{3Kb}{4}$

For slab C, $\beta \equiv \frac{4K(2L \times b)}{4L} = 2Kb$

For slab D, $\beta = \frac{5K(L \times b)}{4L} = \frac{5Kb}{4}$

Clearly, $\frac{3Kb}{4} + \frac{5Kb}{4} = 2Kb.$

$\Rightarrow$ heat flow through slab B + heat flow through slab D = heat flow through C.

The quantity $\frac{l}{KA}$ is called thermal resistance. For a given heat flow rate, the temperature difference is smallest if thermal resistance is smallest.

For slab A thermal resistance is $\frac{1}{8}\alpha$ (α is some constant).

For slabs ($B + C + D$) it is $\frac{1}{4}\alpha$.

For slab E it is $\frac{\alpha}{24}$.

For slab E the thermal resistance, hence the temperature difference across it, is the smallest.

---

6. Four point charges, each of $+q$, are rigidly fixed at the four corners of a square planar soap film of side 'a'. The surface tension of the soap film is γ. The system of charges and planar film are in equilibrium, and $a = k\left[\frac{q^2}{\gamma}\right]^{1/N}$, where '$k$' is a constant. Then N is

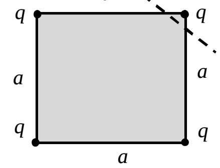
(IIT-JEE-2011, 4 Marks)

Solution: This question defies every conceivable logic.

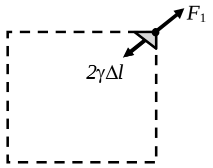
There is a fundamental flaw in framing this problem.

It has been given in the question that the point charges are rigidly fixed at the corners of a square planar soap film. *How*? How do you rigidly fix a charged particle to a liquid film.

Anyway, let us go by what examiner says and assume that somehow it has been done. The square planar soap film of side a with point charge $+q$ rigidly fixed at each of its corners is in equilibrium. See the figure below.

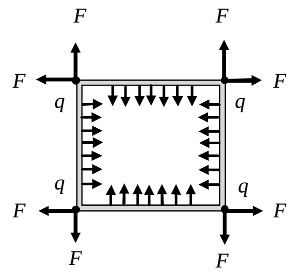


Cut the film along the dotted line shown. Draw the forces on the point charge and the small part of the soap film with it.



The electrostatic repulsion F_1 on the charge has a finite value. The surface tension force on the small part of the film is infinitesimally small. We have the liberty to make Δl as small as we please. Can this part of the system ever be in equilibrium?. How can the net force on this system be zero?

Now consider another system: the four charges and a very thin thread of the soap film. Forces on this system are shown in the figure. The columbic force on each charge has been shown in two components.

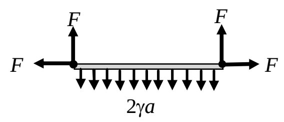


Total surface tension force on each side = $2\gamma a$.

One can say that if $2F = 2\gamma a$, the above system is in equilibrium.

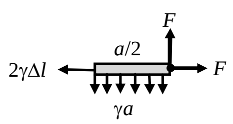
Surely, it is. Don't you see an elephant hanging from a thin cotton thread in this system, tension in the thread being equal to the weight of the elephant!

Consider another system of two point charges and a thread of the soap film.



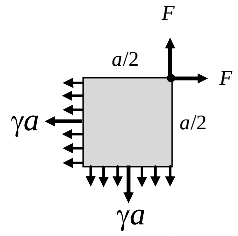
One can say that if $2F = 2\gamma a$ net force on the system is zero, it is in equilibrium. But what about the forces which act along the thread of the soap film? Will they not break it apart?

The following system is still a better choice:

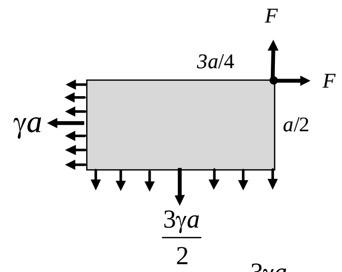


$F = \gamma a$, and $F = 2\gamma\Delta l$ too?

You may be tempted to consider a finite part of the film. Two such systems are given below.



$F = \gamma a$. Fine!

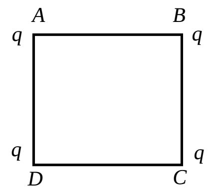


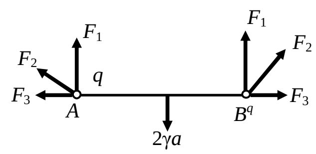
$F = \gamma a$, and $F = \dfrac{3\gamma a}{2}$ too?

Can you now find the mistake in the framing of the problem? Possibly yes.

Now let us solve the problem the way examiner wanted us to do it.

We take the line AB and equate the net vertical force on AB to zero.





$$F_1 = \frac{q^2}{4\pi\,\varepsilon_0\,a^2} = \frac{k'q^2}{a^2}$$

$$F_2 = \frac{q^2}{4\pi\,\varepsilon_0\,(\sqrt{2}a)^2} = \frac{k'q^2}{2a^2}$$

Hence, $2F_1 + 2F_2 \cos\dfrac{\pi}{4} = 2\gamma a$

or $\quad \dfrac{2k'q^2}{a^2} + \dfrac{\sqrt{2}k'q^2}{2a^2} = 2\gamma a$

or $\quad a^3 = \dfrac{q^2 k'\left(1 + \dfrac{1}{2\sqrt{2}}\right)}{\gamma} = \dfrac{k^3 q^2}{\gamma}$, where $k^3 = k'\left(1+\dfrac{1}{2\sqrt{2}}\right)$

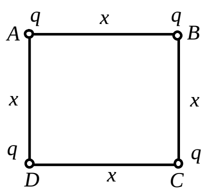
$$\Rightarrow a = k\left[\frac{q^2}{\gamma}\right]^{1/3} \Rightarrow N = 3.$$

A smart idea: If you read the problem carefully, you can see that the value of N can be found out without doing any calculations. The problem has been worded in such a way that you can arrive at the answer without calculating the coulomb force on the charges. From Coulomb's law and principle of superposition you can write that the Coulomb's force on a charge, F_c = (a constant) $\dfrac{q^2}{a^2}$.

Just write this constant as $2k^3$. Then $\dfrac{2k^3q^2}{a^2} = 2\gamma a$, which gives $a = k\left[\dfrac{q^2}{\gamma}\right]^{1/3}$.

I am sure that many students have thought of this and have done this problem in exactly this way in the examination hall. Many brilliant students might have wasted precious exam time in figuring out the problem. Those who are in the habit of doing things correctly may have been victims of great confusion and perplexity.

Second Solution: Let x be the side of the square as shown in the figure below.



For the equilibrium condition the electrostatic potential energy + surface energy must be a minimum. Let the equilibrium be attained at $x = a$.

Now, the total energy,

$$E = 4\cdot\frac{q^2}{4\pi\varepsilon_0 x} + 2\cdot\frac{q^2}{4\pi\varepsilon_0\sqrt{2}x} + \gamma(2x^2)$$

$$= \frac{(4+\sqrt{2})}{4\pi\varepsilon_0}\frac{q^2}{x} + 2\gamma x^2.$$

For equilibrium $\dfrac{d(E)}{dx} = 0$,

or $$-\frac{(4+\sqrt{2})}{4\pi\varepsilon_0}\left(\frac{q^2}{x^2}\right) + 4\gamma x = 0$$

or $$x^3 = \frac{\left(1+\dfrac{1}{2\sqrt{2}}\right)q^2}{4\pi\varepsilon_0\gamma} \equiv \frac{k^3q^2}{\gamma}$$

$$\Rightarrow x = k\left[\frac{q^2}{\gamma}\right]^{1/3}.$$

Since the equilibrium is attained at $x = a$,

$a = k\left[\dfrac{q^2}{\gamma}\right]^{\frac{1}{3}}$. Hence $N = 3$.

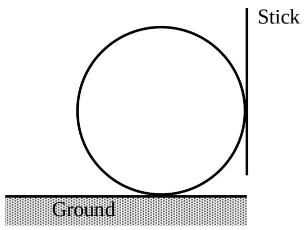
A smart idea again: You can arrive at the answer smartly without doing any calculations. Write the total mechanical energy assuming that the side of the square is x as

$$E = C\frac{q^2}{x} + 2x^2\gamma$$

Minimize E for x. Let the equilibrium be attained at $x = a$. Replace the constant term by k intelligently.

---

7. A boy is pushing a ring of mass 2 kg and radius 0.5 m with a stick as shown in the figure. The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s^2. The coefficient of friction between the ground and the ring is large enough that rolling always occurs and the coefficient of friction between the stick and the ring is (P/10). The value of P is



(IIT-JEE 2011, 4 Marks)

Solution: As per official IIT-JEE website answer of this question is $\boldsymbol{P}$ **= 4**. It was written explicitly in the paper that answer is a single digit integer ranging from 0 to 9. Students faced great difficulty while solving this problem.

'The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s^2'. This statement is not entirely incorrect, but it is certainly confusing. What does the examiner mean by 'the stick applies a force of 2 N on the ring'? Is it the normal force on the ring by the stick or the total force on the ring by the stick? In the first sentence of the question it is given that the boy is pushing the ring. On this basis it can be said that 2 N is the normal force acting on the ring by the stick, not the total force.

Let the coefficient of friction between the stick and the ring be μ. The forces acting on the ring are:

1. 2-N normal force by the stick.

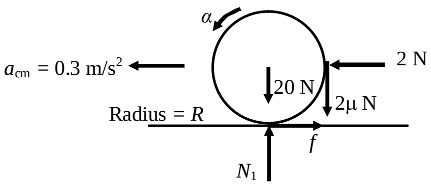
2. Frictional force of value 2μN vertically downward. Note that this frictional force is kinetic in nature. As the ring rolls on the ground, it rotates about its centre of mass in anticlockwise sense. The point of contact of the ring with stick is in motion. The frictional force is opposite to the velocity of point of contact.

3. Weight of the ring 2×10 N vertically downward acting at the centre of gravity.

4. Normal force N_1 at the point of contact by the ground.

5. Frictional force f at the point of contact with the ground. This friction is static in nature. Since the ring always rolls on the ground, the frictional force on the ring by the ground is static friction less than its limiting value. The direction of this friction is towards the right. Why? The tendency of motion of the point of contact is towards left. This is due to the action of 2-N force.

The free body diagram of the ring is as shown in the figure below.



The force and torque equations for the ring are

$2 - f = 2 \times a_{cm} = 2 \times 0.3$ …(1)

$N_1 - 20 - 2\mu = 0$ …(2)

$f \times 0.5 - 2\mu \times 0.5 = 2 \times (0.5)^2 \times \alpha$ …(3)

The constraint equation is

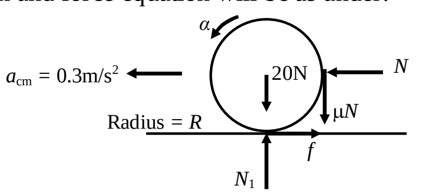
$a_{cm} = R\,\alpha$ or $0.3 = 0.5 \times \alpha$ …(4)

From these equations

$\mu = 0.4$

So, $\frac{P}{10} = 4.0$ or $P = 4$.

But one can also read the problem in a different way. One can assume that 2-N force is the total force acting on the ring by the stick. In that situation the free body diagram and force equation will be as under.



Radius of the ring $R = 0.5$ m.

Acceleration of center of mass, $a_{cm} = 0.3$ m/s^2.

Since the ring rolls on the ground, its angular acceleration $\alpha = \frac{\alpha_{cm}}{R} = \frac{0.3}{0.5} = \frac{3}{5}$ radian/s^2.

Now the force and torque equations are

$N - f = m\,a_{cm} = 2 \times 0.3 = 0.6$ ...(1)

$f\,R - \mu NR = I_{cm}\,\alpha$

or $f \times 0.5 - \mu N \times 0.5 = 2 \times (0.5)^2 \times \frac{3}{5}$

or $f - \mu N = 0.6$ …(2)

From equations (1) and (2),

$$N = \frac{1.2}{1-\mu}.$$

Now if the total force on the ring by the stick is 2 N,

$$\sqrt{N^2 + (\mu N)^2} = 2$$

or $\sqrt{\left(\frac{1.2}{1-\mu}\right)^2 + \left(\frac{1.2\mu}{1-\mu}\right)^2} = 2$

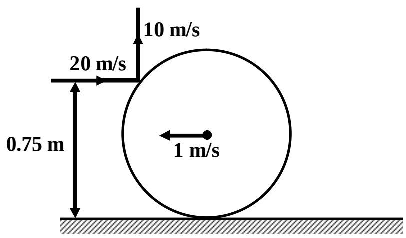
or $\mu = 0.361, 2.763$.

$\Rightarrow$ P = 3.61, 27.63, which is not a SINGLE-DIGIT INTEGER.

Many students wasted lots of precious exam time on this problem. They took the 2 N force as the total force on the ring by the stick.

---

8. A thin ring of mass 2 kg and radius 0.5 m is rolling without slipping on a horizontal plane with velocity 1 m/s. A small ball of mass 0.1 kg, moving with velocity 20 m/s in the opposite direction, hits the ring at a height of 0.75 m and goes vertically up with velocity 10 m/s. Immediately after the collision,



(A) the ring has pure rotation about its stationary CM.

(B) the ring comes to a complete stop.

(C) friction between the ring and the ground is to the left.

(D) there is no friction between the ring and the ground.

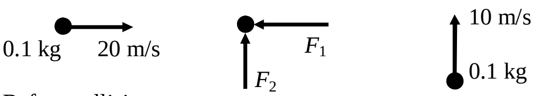
(IIT-JEE 2011 3 Marks)

Solution: The answer is **A** or **A,C** as per official IIT-JEE key. The question created a lot of confusion and stir in the country. The examiner took great care in choosing the values of masses, velocities and lengths. He faltered in phrasing choice (C). Probably, to make up for his misdoings, IIT-JEE organizing body came with two answers: A or AC. But a rigorous analysis shows that even choice (A) cannot be correct.

The examiner did not consider the frictional force on the ring by the ground during the collision. The values of masses and velocities and the answer given on IIT-JEE site corroborate this fact. While solving this problem if you do not consider the frictional force on the ring by the ground then you can show that just after the collision the velocity of centre of mass of the ring is zero; but the ring has an angular velocity about the centre of mass and the frictional force on the ring by the ground is toward left. But the examiner somewhat mis-worded it: *the friction between the ring and ground is to the left.*

In realty, a frictional force, impulsive in nature, that is, of a relatively large value, acts on the ring during the collision. Whether this kind of frictional force is to be considered or not created a great deal of confusion.

To begin with, we consider the examiner's point of view. First analyze the motion of the ball. Suppose the forces acting on the ball are as shown in the figure.



Before collision   Forces on ball during collision   After collision

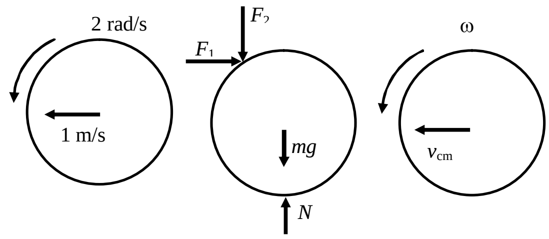
If the time of collision is Δt,

$F_1 \cdot \Delta t = 0.1 \times 20 = 2$ N·s …(1)

$F_2 \cdot \Delta t = 0.1 \times 10 = 1$ N·s …(2)

(Please take care of signs and directions.)

Next consider the ring. Ignoring the frictional force during the collision, the forces on the ring are as shown in the figure:



Before collision   During collision   After collision

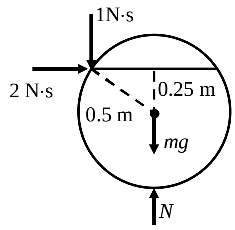
Linear impulse = change in linear momentum gives

$-F_1\,\Delta t = 2\,v_{cm} - 2 \times 1$

or $-2 = 2\,v_{cm} - 2$ (From (1))

or $v_{cm} = 0$.

And angular impulse about CM = change in angular momentum about CM gives,

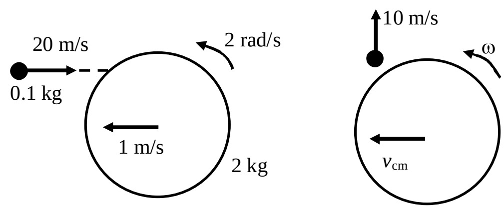


$$1 \times 0.5 \times \frac{\sqrt{3}}{2} - 2 \times 0.25 = 2 \times (0.5)^2(\omega - 2)$$

or $\omega = 1.866$ rad/s anticlockwise.

That is, immediately after collision, $v_{cm} = 0$ and $\omega = 1.866$ rad/s anticlockwise. The ring rotates about stationary centre of mass. Velocity of the point of contact with ground is to the right. The friction on the ring by the ground is to the left.

You can also arrive at these answers by conserving linear momentum, and angular momentum of the ring ball system relative to the ground.



Just before collision   Just after collision

Conservation of linear momentum

$-0.1 \times 20 + 2 \times 1 = 2 \times v_{cm} \Rightarrow v_{cm} = 0$.

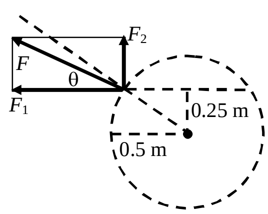
Conservation of angular momentum relative to the ground

$$\underbrace{2 \times 1 \times 0.5}_{M\,v_{cm}\,r} + \underbrace{2 \times (0.5)^2 \times 2}_{I_{cm}\,\omega_1} - \underbrace{0.1 \times 20 \times 0.75}_{mv_i r_\perp}$$

$$= \underbrace{2 \times (0.5)^2\,\omega}_{I_{cm}\,\omega_2} - \underbrace{0.1 \times 10 \times 0.5 \times \frac{\sqrt{3}}{2}}_{m\,v_f\,r_\perp}$$

$$\Rightarrow \omega = 1 + \frac{\sqrt{3}}{2} = 1.866 \text{ rad/s.}$$

Now let us check whether the force on the ball by the ring is normal to the ring or not. The situation is shown in the figure below.



From the above figure,

$\tan\theta = \dfrac{F_2}{F_1} = \dfrac{1}{2}.$ From equations (1) and (2)

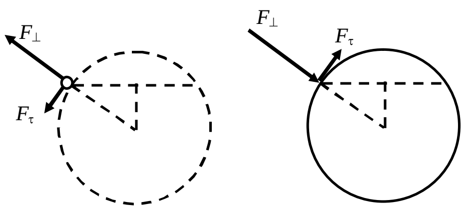
$\Rightarrow \theta < 30°.\ \left(\tan\theta = \dfrac{1}{2}, \tan 30^\circ = \dfrac{1}{\sqrt{3}} \text{ and } 2 > \sqrt{3}\right)$

Hence the total force on the ball is not normal to the ring. Further let us resolve the resultant force into components:

(1) Normal component $F_\perp = F \cos(30° - \theta)$, the component which is normal to the ring.

(2) Tangential component $F_\tau = F \sin(30° - \theta)$, the component which is tangential to the ring.

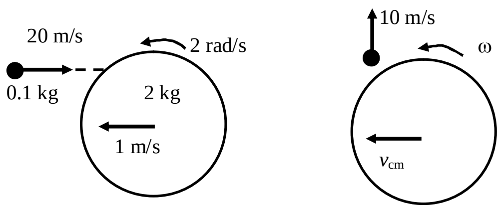
These two components and their reactions are shown in the figures below.



Force on ball by ring   Force on ring by ball

From the above figure it can be inferred that during the collision there is a frictional force $F_\tau = F \sin(30° - \theta)$ on the ball by the ring. The reaction of this force acts on the ring. If the ball applies a frictional force on the ring during the collision, so does the ground. It can be shown that the point of contact of the ring has a tendency of motion relative to the ground.

Hence one may expect a frictional force on the ring by the ground. Moreover, during the collision, the normal force on the ring by the ground takes a large value. This can be inferred from the free body diagram of the ring.



Before collision   After collision

From conservation of linear momentum,

$2 \times 1 \times 0.5 + 2 \times (0.5)^2 \times 2 - 1.0 \times 20 - 0.75$

$= 2 \times v_{cm} \times 0.5 + 2 \times (0.5)^2\,\omega - 0.1 \times 10 \times 0.5 \times \dfrac{\sqrt{3}}{2}.$

From this single equation you cannot find out two unknown quantities v_{cm} and ω. Therefore, you cannot really conclude anything.

So the most important point is whether to consider the impulsive friction or not. The examiner (probably) did not consider it.

Random Problems from ‘The Art of problem Solving in Physics’ Volume : I

1. A cannon fires successively two shells with velocity ; $v_0 = 250$ m/s , the first at the angle $\theta_1 = 60^\circ$ and the second at the angle $\theta_2 = 45^\circ$ to the horizontal, the azimuth being the same. Neglecting the air drag, find the time interval between firings leading to the collision of the shells

Solution: Let Shell-*1* be projected from origin with velocity v_0 at angle θ_1 to the horizontal at moment $t = 0,$ and Shell-*2* with same velocity v_0 at angle θ_2 at the moment $t = \tau$. Let them collide at moment $t = t$ at point $(x,\ y)$. Shell-*1*, clearly, flies for time *t* and Shell-2 for time $(t - \tau)$ before they collide.

Motion along *x*-axis:

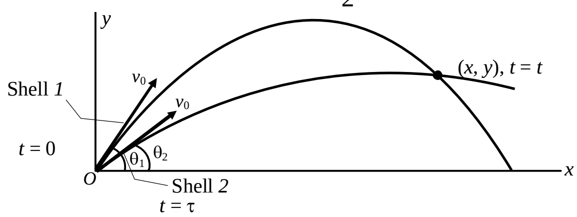
Shell-*1*: $\quad x = v_0 \cos\theta_1 \cdot t \qquad \ldots(1)$

Shell-*2*: $\quad x = v_0 \cos\theta_2 \cdot (t - \tau) \qquad \ldots(2)$

Motion along *y*-axis:

Shell-*1*: $\quad y = v_0 \sin\theta_1 \cdot t - \dfrac{1}{2} g t^2 \qquad \ldots(3)$

Shell-*2*: $\quad y = v_0 \sin\theta_2 \cdot (t - \tau) - \dfrac{1}{2} g\,(t - \tau)^2 \qquad \ldots(4)$



From equations (1) and (2)

$$v_0 \cos\theta_1 \cdot t = v_0 \cos\theta_2 \cdot (t - \tau)$$

or $\quad t = \dfrac{\cos\theta_2 \cdot \tau}{\cos\theta_2 - \cos\theta_1}. \qquad \ldots(5)$

And from equations (3) and (4),

$$v_0 \sin\theta_1 \cdot t - \frac{1}{2} g t^2 = v_0 \sin\theta_2 \cdot (t - \tau) - \frac{1}{2} g (t - \tau)^2 \quad \ldots(6)$$

Substituting $t = \dfrac{\cos\theta_2 \cdot \tau}{\cos\theta_2 - \cos\theta_1}$ from equation (5) in equation (6),

$$v_0 \sin\theta_1 \left(\frac{\cos\theta_2 \cdot \tau}{\cos\theta_2 - \cos\theta_1} \right) - \frac{1}{2} g \left(\frac{\cos\theta_2 \cdot \tau}{\cos\theta_2 - \cos\theta_1} \right)^2$$

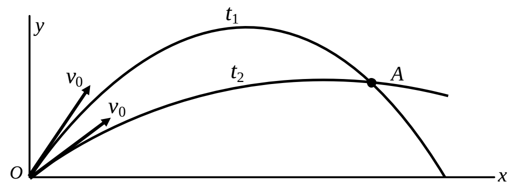
$$= v_0 \sin\theta_2 \cdot \left(\frac{\cos\theta_2 \cdot \tau}{\cos\theta_2 - \cos\theta_1} - \tau \right) - \frac{1}{2} g \left(\frac{\cos\theta_2 \cdot \tau}{\cos\theta_2 - \cos\theta_1} - \tau \right)^2 .$$

Solve this equation for τ.

$$\tau = \frac{2v_0}{g} \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1 + \cos\theta_2}.$$

Second Solution: Let the Shell-*1* fly for time t_1 and Shell-*2* for time t_2 before colliding at point *A*, as shown in the figure. During these times t_1 and t_2, the shells undergo same horizontal and vertical displacements. That is,

$v_0 \cos\theta_1 \cdot t_1 = v_0 \cos\theta_2 \cdot t_2 \qquad \ldots(1)$



and $\quad v_0 \sin\theta_1 \cdot t_1 - \dfrac{1}{2} g t_1^2 = v_0 \sin\theta_2 \cdot t_2 - \dfrac{1}{2} g t_2^2 \qquad \ldots(2)$

On dividing equation (2) by equation (1), you get,

$$\frac{v_0 \sin\theta_1 - \frac{1}{2} g t_1}{v_0 \cos\theta_1} = \frac{v_0 \sin\theta_2 - \frac{1}{2} g t_2}{v_0 \cos\theta_2},$$

which gives

$\dfrac{2v_0}{g} \sin(\theta_1 - \theta_2) = t_1 \cos\theta_2 - t_2 \cos\theta_1 \qquad \ldots(3)$

On solving equations (1) and (3), you get

$$t_1 = \frac{2v_0}{g} \cdot \sin(\theta_1 - \theta_2) \cdot \frac{\cos\theta_2}{\cos^2\theta_2 - \cos^2\theta_1}.$$

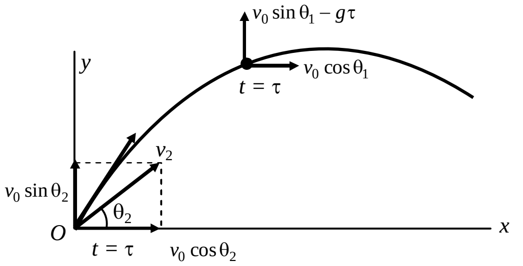
Required $\Delta t = t_1 - t_2 = t_1 - t_1 \dfrac{\cos\theta_1}{\cos\theta_2}$

$$= t_1 \cdot \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_2}$$

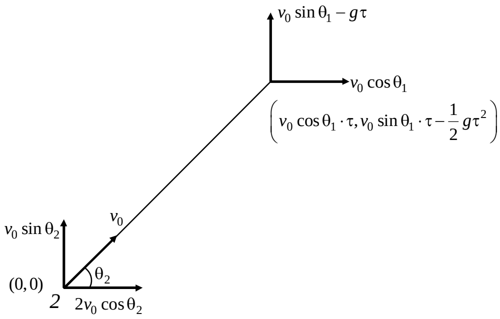
$$= \frac{2v_0}{g} \sin(\theta_1 - \theta_2) \cdot \frac{\cos\theta_2}{\cos^2\theta_2 - \cos^2\theta_1} \cdot \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_2}$$

$$= \frac{2v_0}{g} \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1 + \cos\theta_2}.$$

Third Solution: Shell-*1* is thrown at moment $t = 0,$ with velocity v_0 at angle θ_1 to the horizontal. And at moment $t = \tau,$ Shell-*2* is thrown from the same point with velocity v_0 at angle θ_2 to the horizontal. Refer to the coordinate system given in the figure below.



At the moment $t = \tau$, the Shell-1 is at point $\left(v_0 \cos\theta_1 \cdot \tau,\ v_0 \sin\theta_1 \cdot \tau - \frac{1}{2} g\tau^2 \right)$ and has *x*- and *y*-components of velocity as $v_0 \cos\theta_1$ and $v_0 \sin\theta_1 - g\tau$ respectively. Figure below depicts the positions and velocities of Shell-*1* and Shell-*2* at moment $t = \tau$.



Let the game start at the moment $t = \tau$. This moment onwards acceleration of Shell-*2* with respect to Shell-*1* is zero. How? If you look at Shell-*2* from Shell-*1*, its velocity must be aimed at Shell-*1* for the shells to collide. That is, the direction of velocity of Shell-*2* *wrt* Shell-*1* $\vec{v}_{21}$ must be in the direction of vector $\overrightarrow{21}$. From the figure,

$$\vec{v}_{21} = (v_0 \cos\theta_2 - v_0 \cos\theta_1) i + (v_0 \sin\theta_2 - v_0 \sin\theta_1 + g\tau) j$$

and $\overrightarrow{21} = (v_0 \cos\theta_1 \cdot \tau) i + \left(v_0 \sin\theta_1 \cdot \tau - \frac{1}{2} g\tau^2 \right) j$

Since the vector $\vec{v}_{21}$ is in the direction of vector $\overrightarrow{21}$,

$$\frac{v_0 \cos\theta_2 - v_0 \cos\theta_1}{v_0 \cos\theta_1 \cdot \tau} = \frac{v_0 \sin\theta_2 - v_0 \sin\theta_1 + g\tau}{v_0 \sin\theta_1 \cdot \tau - \frac{1}{2} g\tau^2},$$

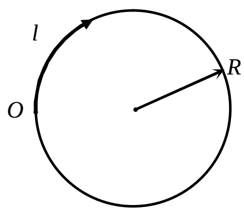
which on solving for τ gives $\tau = \frac{2v_0}{g} \cdot \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1 + \cos\theta_2}$.

2. A particle moves along an arc of a circle of radius *R* according to the law $l = a \sin \omega t$, where *l* is the displacement from the initial position measured along the arc, and *a* and ω are constants. Find:
(a) the magnitude of the total acceleration of the particle at the points $l = 0$ and $l = \pm a$;
(b) the minimum value of the total acceleration $(a_{\text{Total}})_{\min}$ and the corresponding displacement l_m.

Solution: (a) The motion of the particle is depicted in the figure below.

Since l is displacement from the initial position measured along the arc, velocity *v* is given by $v = \frac{dl}{dt} = a\omega \cos \omega t$, and tangential acceleration is given by $a_\tau = \frac{dv}{dt} = -a\omega^2 \sin\omega t.$

The radial acceleration



$$a_r = \frac{v^2}{R} = \frac{(a\omega \cos\omega t)^2}{R}$$

$$= \frac{a^2\omega^2 \cos^2 \omega t}{R}$$

Now total acceleration

$$a_{\text{Total}} = \sqrt{{a_\tau}^2 + {a_r}^2}$$

$$= \sqrt{a^2\omega^4 \sin^2 \omega t + \left(\frac{a^2\omega^2 \cos^2 \omega t}{R} \right)^2}$$

For $l = 0$, $a \sin \omega t = 0$ or $\omega t = 0$, and so $a_{\text{Total}} = \frac{a^2\omega^2}{R}$.

For $l = \pm a$, $\omega t = \pm \frac{\pi}{2}$, and for this value of ωt, $a_{\text{Total}} = a\omega^2$.

(b) $a_{\text{Total}} = \sqrt{\left(\frac{a^2\omega^2 \cos^2 \omega t}{R} \right)^2 + a^2\omega^4 \sin^2 \omega t}$ …(1)

Let $f = \frac{a^4\omega^4 \cos^4 \omega t}{R^2} + a^2\omega^4 \sin^2 \omega t$.

$\frac{df}{dt} = 0$ gives

$$-\frac{a^4\omega^4}{R^2} \cdot 4\cos^3 \omega t \sin\omega t \cdot \omega + a^2\omega^4 \cdot 2\sin\omega t \cos\omega t \cdot \omega = 0$$

or $\cos^2 \omega t = \frac{R^2}{2a^2}$.

For the values of t given by the above equation a_{Total} will take its minimum value (Why?). On putting $\cos^2 \omega t = \frac{R^2}{2a^2}$ in equation (1),

$$(a_{\text{Total}})_{\min} = a\omega^2 \sqrt{1 - \left(\frac{R}{2a} \right)^2}.$$

Second Solution:

$$a_{\text{Total}} = \sqrt{\left(\frac{a^2\omega^2 \cos^2 \omega t}{R} \right)^2 + (a\omega^2 \sin\omega t)^2}$$

$$= a\omega^2 \sqrt{\frac{a^2}{R^2} \cos^4 \omega t + (1 - \cos^2 \omega t)}$$

$$= a\omega^2\sqrt{1-\left(\underbrace{\left(\frac{a^2}{R^2}\cos^2\omega t\right)\left(1-\frac{a^2}{R^2}\cos^2\omega t\right)}_{y}\right)\frac{R^2}{a^2}}$$

It can be seen from the above equation that a_{Total} will take its minimum value when y is maximum.

1. From $AM \geq GM$,

$$\frac{\left(\frac{a^2}{R^2}\cos^2\omega t\right)+\left(1-\frac{a^2}{R^2}\cos^2\omega t\right)}{2} \geq \sqrt{y}$$

or $\quad y \leq \frac{1}{4}$

$\Rightarrow \quad y_{\max} = \frac{1}{4}$.

Then $(a_{\text{Total}})_{\min} = a\omega^2\sqrt{1-\frac{1}{4}\frac{R^2}{a^2}} = a\omega^2\sqrt{1-\left(\frac{R}{2a}\right)^2}$.

2. The term y is maximum when

$\frac{a^2}{R^2}\cos^2\omega t = 1-\frac{a^2}{R^2}\cos^2\omega t$. Why?

($p+q=c$, then $p \cdot q$ is maximum when $p=q=\frac{c}{2}$.)

Since $\left(\frac{a^2}{R^2}\cos^2\omega t\right)+\left(1-\frac{a^2}{R^2}\cos^2\omega t\right)=1$,

$\left(\frac{a^2}{R^2}\cos^2\omega t\right)\cdot\left(1-\frac{a^2}{R^2}\cos^2\omega t\right)$ will be maximum when

$$\frac{a^2}{R^2}\cos^2\omega t = 1-\frac{a^2}{R^2}\cos^2\omega t\left(=\frac{1}{2}\right).$$

Therefore,

$$(a_{\text{Total}})_{\min} = a\omega^2\sqrt{1-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{R^2}{a^2}} = a\omega^2\sqrt{1-\left(\frac{R}{2a}\right)^2}.$$

Third Solution:

$$a_{\text{Total}} = \sqrt{\left(\frac{a^2\omega^2\cos^2\omega t}{R}\right)+(a\omega^2\sin\omega t)^2}$$

$$= \sqrt{\frac{a^2\omega^2\cos^2\omega t}{R}+a^2\omega^4(1-\cos^2\omega t)}$$

$$= \frac{a\omega^2}{R}\sqrt{a^2\cos^4\omega t - R^2\cos^2\omega t + R^2}.$$

The quadratic expression in $\cos^2\omega t$ under the square root sign will take its minimum value when $\cos^2\omega t = \frac{R^2}{2a^2}$.

(If $a > 0$, then the expression ax^2+bx+c takes its minimum value when $x = -\frac{b}{2a}$.)

3. A small block A is placed on an inclined plane forming an angle α with the horizontal, see the figure and is imparted an initial velocity v_0. Find how the velocity of the block depends on angle ϕ if the coefficient of friction $\mu = \tan\alpha$ and at the initial moment $\phi_0 = \pi/2$.

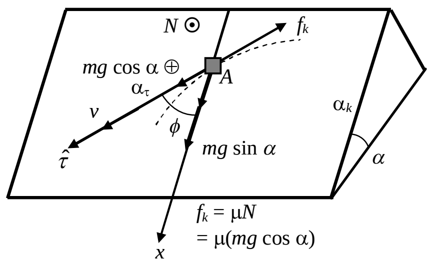
Solution: How to (rather from where to) get started? This is the only problem in this question. Even after having drawn the free-body diagram correctly you will not be sure of how to apply $\Sigma\vec{F} = m\vec{a}$ to the block, how to set the coordinate system!

The forces on block A sliding on the incline are-

(i) The weight mg acting vertically downward. Shown in the figure are two components of the $mg - mg \sin\alpha$ down the incline, along the line of maximum slope and $mg\cos\alpha$, normal to the incline shown as $mg\cos\alpha \otimes$.

(ii) The normal force, N, normal to the incline, shown as $N \otimes$. Note that $N = mg\cos\alpha$. The block moves along the incline. It does not have any acceleration normal to the incline.

(iii) The frictional force $f_k = \mu_k N = \mu_k mg\cos\alpha$. This force is parallel to the inclined plank in opposite direction of velocity of the block. Velocity of the block is along the incline tangent to the path it follows on it. At moment t shown in the figure, velocity of the block makes angle ϕ with the line of maximum slope at that point.



You have to find v in terms of ϕ. Should you form a differential equation in terms of v and ϕ? No. Why?

You try to do it to get the answer of "why". Should you try to relate a_τ, a_x and ϕ ? Many of you will do the mistake of writing that $a_x = a_\tau \cos\phi$. Why is $a_x = a_\tau \cos\phi$ not correct? If you can answer this question without any calculation whatsoever your fundamentals are OK. What next? Clearly $v_x = v\cos\phi$, and $\frac{dv_x}{dt} = a_x$, $\frac{dv}{dt} = a_\tau$. So try to relate a_x and a_τ. But how to do it?

Just write force equations along the directions $\hat{\tau}$ and $\hat{x}$.

From $\Sigma F_\tau = ma_\tau$.

$$mg\sin\alpha\cos\phi - mg\sin\alpha = ma_\tau \quad \ldots (1)$$

And from $\Sigma F_x = ma_x$,

$$mg\sin\alpha - mg\sin\alpha\cos\phi = ma_x \quad \ldots (2)$$

From equations (1) and (2),

$a_\tau = -a_x$ … (3) [So simple!]

or $\quad \frac{dv}{dt} = -\frac{dv_x}{dt} \qquad$ or $\qquad dv = -\,dv_x$

or $\int dv = -\int dv_x$ or $v = -v_x + c$ … (4)

Now when $v = v_0$, $v_x = 0$. [$\because$ initially $\phi = \pi/2$, v_0 is perpendicular to x-axis]

or $v_0 = 0 + c$

or $c = v_0$

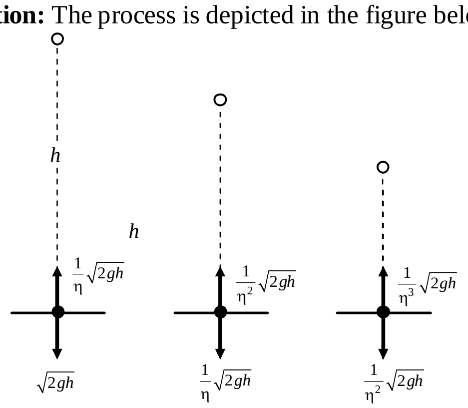
Therefore, $v = -v_x + v_0$

or $v = -v\cos\phi + v_0$ [$\because v_x = v\cos\phi$]

or $v = \dfrac{v_0}{1+\cos\phi}$.

4. A steel ball of mass $m = 50\,g$ falls from the height $h = 1.0$ m on the horizontal surface of a massive slab. Find the cumulative momentum that the ball imparts to the slab after numerous bounces, if every impact decreases the velocity of the ball $\eta = 1.25$ times.

Solution: The process is depicted in the figure below.



I Impact II Impact III Impact

Change in momentum of the ball during first impact

$$\Delta p_1 = -m\frac{1}{\eta}\sqrt{2gh} - (m\sqrt{2gh}).$$

Here, the downward direction has been taken as positive.

The momentum delivered to the slab during first impact

$$\Delta p_1' = m\frac{1}{\eta}\sqrt{2gh} + m\sqrt{2gh} = m\sqrt{2gh}\left(\frac{1}{\eta}+1\right).$$

Change in the momentum of the ball during second impact

$$\Delta p_2 = -m\frac{1}{\eta^2}\sqrt{2gh} - \left(m\frac{1}{\eta}\sqrt{2gh}\right).$$

The momentum delivered to the slab during second impact

$$\Delta p_2' = m\frac{1}{\eta^2}\sqrt{2gh} + m\frac{1}{\eta}\sqrt{2gh} = m\sqrt{2gh}\left(\frac{1}{\eta^2}+\frac{1}{\eta}\right)$$

Similarly, the momentum delivered to the slab during third impact

$$\Delta p_3' = m\sqrt{2gh}\left(\frac{1}{\eta^3}+\frac{1}{\eta^2}\right).$$

Total momentum delivered to the slab by the ball

$$\Delta p' = \Delta p_1' + \Delta p_2' + \Delta p_3' + \ldots$$

$$= m\sqrt{2gh}\left(\frac{1}{\eta}+1\right) + m\sqrt{2gh}\left(\frac{1}{\eta^2}+\frac{1}{\eta}\right) + m\sqrt{2gh}\left(\frac{1}{\eta^3}+\frac{1}{\eta^2}\right) + \ldots$$

$$= m\sqrt{2gh}\left(1+\frac{2}{\eta}+\frac{2}{\eta^2}+\frac{2}{\eta^3}+\ldots\right) = m\sqrt{2gh}\left(\frac{\eta+1}{\eta-1}\right)$$

$$= \frac{50}{1000}\cdot\sqrt{2\times 9.8\times 1.0}\left(\frac{1.25+1}{1.25-1}\right) = 0.2\,\text{kg m/s}.$$

Second Solution: The ball hits the slab for the first time with velocity $\sqrt{2gh}$ after time $t_1 = \sqrt{\dfrac{2h}{g}}$. Then it rebounds with velocity $\dfrac{\sqrt{2gh}}{\eta}$, goes up for time t_2 and then comes down in time t_2 to hit the slab for the second time, where $t_2 = \dfrac{1}{\eta}\sqrt{\dfrac{2h}{g}}$. After the second impact the ball bounces with speed $\dfrac{1}{\eta^2}\sqrt{2gh}$, it goes up for time t_3 and then comes down in time t_3 to hit the slab for the third time, here $t_3 = \dfrac{1}{\eta^2}\sqrt{\dfrac{2h}{g}}$.

The ball is in air for a total time

$t = t_1 + 2t_2 + 2t_3 + \ldots$ ad infinitum

$$= \sqrt{\frac{2h}{g}} + 2\frac{1}{\eta}\sqrt{\frac{2h}{g}} + 2\frac{1}{\eta^2}\sqrt{\frac{2h}{g}} + \ldots$$

$$= \sqrt{\frac{2h}{g}}\left(\frac{\eta+1}{\eta-1}\right).$$

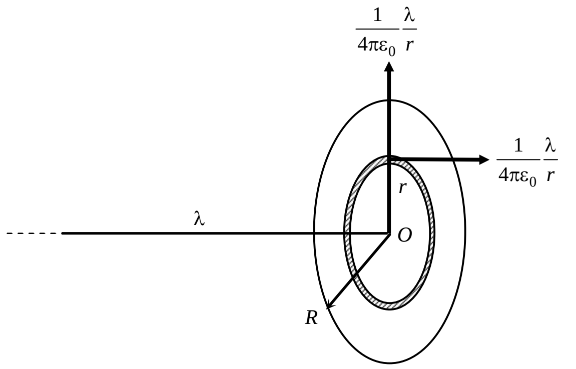
During this time t, the gravity force imparts a total momentum mgt to the ball. Eventually the ball comes to rest. So it delivers to the slab all the momentum it acquires during its motion.

Therefore, the cumulative linear momentum that ball imparts to the slab $= mg\,t = m\sqrt{2gh}\left(\dfrac{\eta+1}{\eta-1}\right)$.

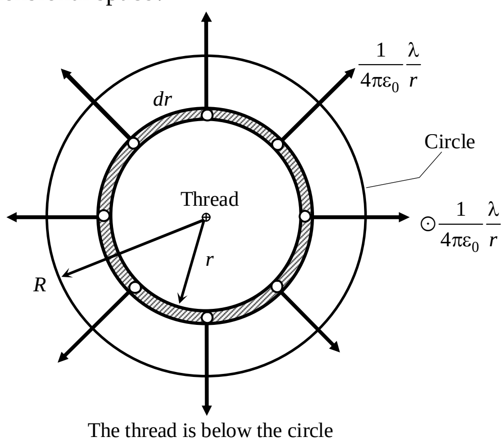
Random Problems from 'The Art of problem Solving in Physics' Volume : II

1. A very long uniformly charged thread oriented along the axis of a circle of radius R rests on its centre with one of the ends. The charge of the thread per unit length is equal to λ. Find the flux of the vector $\vec{E}$ across the circle area.

Solution: The expression for the electric field at a point at a distance r on the perpendicular passing through one of the ends of a long thread (calculated in Problem 3.14) can be used to find the flux through the circle in a very simple way.



The component of the electric field which is parallel to the thread is normal to the circle and the component which is perpendicular to the thread is parallel to the circle. Are you comfortable in visualizing this in three dimensional space?



The thread is below the circle passing though its centre

Which component of the electric field contributes to flux? The one that is perpendicular to the circle or the one parallel to it? How does the component of electric field that is normal to the circle's plane vary with the distance of the point from its centre? Now, finally can you exploit symmetry in computing the integral $\int \vec{E}\cdot d\vec{s}$? From symmetry considerations, flux through a thin strip of radius r and thickness dr is $\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\cdot 2\pi r\, dr$. Whence the total flux through the circle,

$$\phi = \int_0^R \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\cdot 2\pi r\, dr = \frac{\lambda R}{2\varepsilon_0}.$$

Second Solution: The solid angle subtended by a right circular cone on its vertex is $\Omega = 2\pi\,(1 - \cos\alpha)$, where α is half of the cone angle. This proposition can be easily established using the definition of solid angle.

Electric flux through an area dS due to a point charge q is related to the solid angle the area dS subtends at the charge as follows.

$$d\phi = \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\frac{dS\cos\theta}{r^2} = \frac{q\,\Omega}{4\pi\varepsilon_0}\cdot$$

where $d\Omega$ is solid angle subtended by area dS on the point charge.

Consider an element of length dx of the thread at a distance x from centre of circle.

The charge on the element of length dx is $dq = \lambda dx$.

The solid angle subtended by the circle on dq is

$$\Omega = 2\pi\,(1 - \cos\alpha).$$

$$\Omega = 2\pi\left(1 - \frac{x}{\sqrt{R^2 + x^2}}\right).$$

The electric flux passing through the circle due to element charge dq

$$d\phi = \frac{\Omega dq}{4\pi\varepsilon_0} = \frac{2\pi}{4\pi\varepsilon_0}\left(1 - \frac{x}{\sqrt{R^2 + x^2}}\right)\lambda dx$$

Flux through the circle due to the thread of length l,

$$\phi = \frac{\lambda}{2\varepsilon_0}\int_0^l \left(1 - \frac{x}{\sqrt{R^2 + x^2}}\right)dx$$

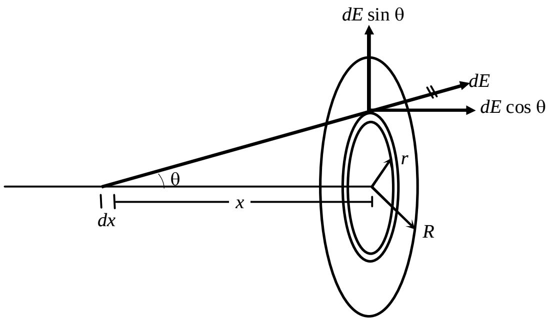
$$= \frac{\lambda}{2\varepsilon_0}\left(l - \left(\sqrt{l^2 + R^2} + R\right)\right).$$

If the thread is long, $\frac{R^2}{l^2} \to 0$, whence

$$\phi = \frac{\lambda l}{2\varepsilon_0}\left(1 - \sqrt{1 + \frac{R^2}{l^2}} + \frac{R}{l}\right)$$

$$= \frac{\lambda R}{2\varepsilon_0}.$$

Third Solution: Nobody remembers the expression for the field calculated in Problem 3.14. Most of you may not be able to apply the idea of solid angle to solve this problem. Suppose, all you remember is the field of a point charge, $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ and definition of flux $d\phi = \vec{E}\cdot d\vec{S}$. If that is the case, how would you solve the problem? It's easy. Consider a point charge on the thread. Find the flux through the circle as if this were the only charge present. But, of course, this is not the only charge! So integrate all over the thread.



Refer to the figure above. Flux through a thin strip of radius r and thickness dr if only $dq = \lambda dx$ were present is $dE\cos\theta\cdot 2\pi r dr$. Integration over the circle gives the flux through it if only charge dq were present.

$$d\phi = \int_0^R \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(x^2+r^2)}\cdot\frac{x}{\sqrt{x^2+r^2}}2\pi r dr.$$

[Note that x is the variable of integration for thread and not for the circle.]

The contribution of the whole thread to the flux through the circle is

$$\phi = \int_0^\infty\int_0^R \frac{1}{4\pi\varepsilon_0}\frac{x\lambda dx}{(x^2+r^2)^{3/2}}2\pi r dr.$$

First integrate over the circle, then over the thread.

$$\int_0^R \frac{rdr}{(x^2+r^2)^{3/2}} = \frac{1}{x}\left(1-\frac{x}{\sqrt{x^2+R^2}}\right).$$

The integration is done by assuming $x^2+r^2=t$ and $2rdr = dt$, here x is a constant. Now substitute $rdr = \frac{dt}{2}$ and $R(x^2+r^2) = t^{3/2}$. Whence

$$\phi = \int_0^\infty \frac{\lambda x}{2\varepsilon_0}\cdot\frac{1}{x}\left(1-\frac{x}{\sqrt{R^2+x^2}}\right)dx$$

$$= \frac{\lambda}{2\varepsilon_0}\lim_{t\to\infty}\left(t-\int_0^t\left(\frac{xdx}{\sqrt{R^2+x^2}}\right)\right)$$

$$= \frac{\lambda}{2\varepsilon_0}\lim_{t\to\infty}\left(t-\sqrt{t^2+R^2}+R\right)$$

$$= \frac{\lambda}{2\varepsilon_0}\lim_{t\to\infty}\left(-\frac{R^2}{t+\sqrt{t^2+R^2}}+R\right)$$

$$= \frac{\lambda}{2\varepsilon_0}\ (0+R)$$

$$= \frac{\lambda R}{2\varepsilon_0}.$$

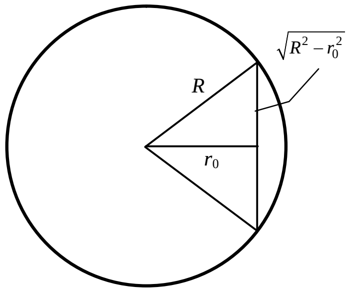
We have used the most basic manipulations of calculus to arrive at the answer.

2. A ball of radius R is uniformly charged with the volume density ρ. Find the flux of the electric field strength vector across the ball's section formed by the plane located at a distance $r_0 < R$ from the centre of the ball.

Solution: This is an absolutely simple problem. All you need to do is to calculate the charge contained in the cone whose vertex is at the centre of the sphere and whose base is the ball's section formed by the plane located at a distance $r_0 < R$ from the centre of the ball.

Apply the formula $V = \frac{\pi r^2 h}{3}$, to obtain the volume of the cone described above. You have,

$$V = \frac{\pi(R^2-r_0^2)\cdot r_0}{3}. \qquad [\because h = r_0]$$



Thus, charge contained in the cone,

$$q = \rho\cdot\frac{1}{3}\pi(R^2-r_0^2)r_0.$$

Whence the required flux,

$$\phi = \frac{q}{\varepsilon_0} = \frac{\rho\cdot\frac{1}{3}\pi\left(R^2-r_0^2\right)r_0}{\varepsilon_0}$$

$$= \frac{\pi\rho}{3\varepsilon_0}r_0(R^2-r_0^2).$$

Aren't you amazed? And you know, most of the problems you are asked to solve are this simple!

Second Solution: Consider a thin strip element of radius $y\left[0\le r\le\sqrt{R^2-r_0^2}\right]$ and thickness dy. Draw a picture in the mind as visualization of the situation is the most important aspect of problem solving. As computed in Problem 3.16(b), electric field at each

point of the ring is $\vec{E} = \dfrac{\rho\vec{r}}{3\varepsilon_0}$ where $\vec{r}$ is the radius vector of the point on the strip from the centre of the sphere. Note that $\vec{r}$ is a variable vector.
The electric flux through this circular strip is

$$d\phi = \frac{\rho r ds \cos\theta}{3\varepsilon_0}$$

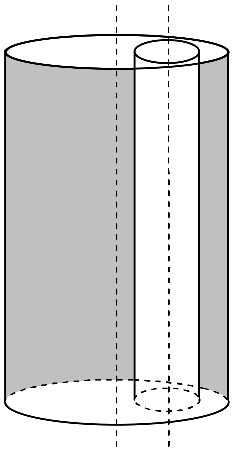
$$= \frac{2\pi\rho r\, y\, dy}{3\varepsilon_0}.$$

Total flux through the circular surface

$$\phi = \int_0^{\sqrt{R^2 - r_0^2}} \frac{2\pi\rho r_0}{3\varepsilon_0} y\, dy$$

$$= \frac{\pi\rho r_0 (R^2 - r_0^2)}{3\varepsilon_0}.$$

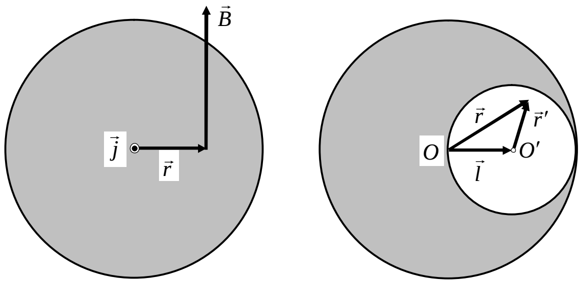
3. Inside a long straight uniform wire of round cross-section there is a long round cylindrical cavity whose axis is parallel to the axis of the wire and displaced from the latter by a distance l. A direct current of density $\vec{j}$ flows along the wire. Find the magnetic induction inside the cavity. Consider, in particular, the case $l = 0$.



Solution. In accordance with the principle of superposition, the required quantity can be represented as follows:

$$\vec{B}_0 = \vec{B}_1 - \vec{B}_2, \qquad \ldots(1)$$

where $\vec{B}_1$ is the magnetic induction of the conductor without cavity, while $\vec{B}_2$ is the magnetic induction of the field at the same point due to the current flowing through the part of the conductor which has been removed in order to create the cavity.
Thus, the problem requires first of all the calculation of magnetic induction $\vec{B}_1$ inside the solid conductor at a distance r from its axis. Using the theorem on circulation, we can write $2\pi r B = \mu_0 \pi r^2 j$, whence $B = \frac{1}{2}\mu_0 r j$. This expression can be represented with the help of the figure in the vector form:



$$\vec{B} = \frac{1}{2}\mu_0[\vec{j} \times \vec{r}].$$

Using now this formula we can write expression for $\vec{B}_1$ and $\vec{B}_2$,

$$\vec{B}_1 = \frac{\mu_0}{2}[\hat{j} \times \vec{r}]$$

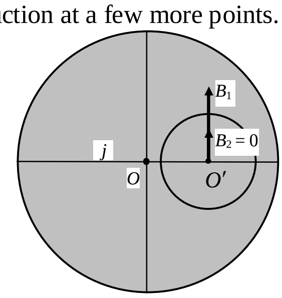
and $\qquad \vec{B}_2 = \dfrac{\mu_0}{2}[\vec{j} \times \vec{r}'].$

On substituting the expressions for $\vec{B}_1$ and $\vec{B}_2$ in equation (1) you get

$$\vec{B}_0 = \frac{\mu_0}{2}[\vec{j} \times \vec{r}] - \frac{\mu_0}{2}[\vec{j} \times \vec{r}'] = \frac{\mu_0}{2}[\vec{j} \times (\vec{r} - \vec{r}')].$$

Figure shows that $\vec{r} = \vec{l} + \vec{r}'$, whence $\vec{r} - \vec{r}' = \vec{l}$, and $\vec{B} = \frac{1}{2}\mu_0[\vec{j} \times \vec{l}]$.

Thus, in our case the induction $\vec{B}$ of the magnetic field in the cavity is uniform, and if the current is flowing towards us see the figure, the field $\vec{B}$ lies in the plane of the figure and is directed as shown.
Additionally, for the sake of clarity and practice we compute the magnetic induction at a few more points.

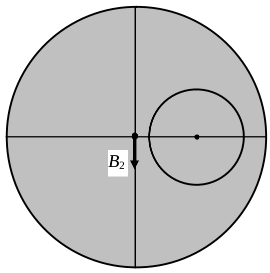


Let the current density $\hat{j}$ be in the sense represented by $\odot$, coming out of the page.
(i) At point on the axis of cavity

$B_1 \cdot 2\pi l = \mu_0 j\pi R^2$

$B_2 = 0$

$B = B_1 - B_2$.



(ii) At the axis of the wire:

$B_1 = 0$

$B_2 \cdot 2\pi l = \mu_0 j\pi r^2$

$B = 0 - B_2$

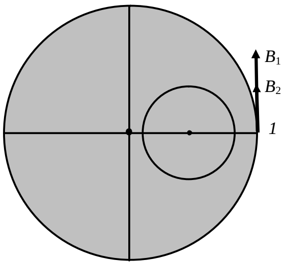
Can you tell the magnitude and direction of B?

(iii) At point *1* in the figure:

$B_1 \cdot 2\pi R = \mu_0 j\pi R^2$

$B_2 \cdot 2\pi(R-l) = \mu_0 j\pi r^2$

$B = B_1 - B_2$.

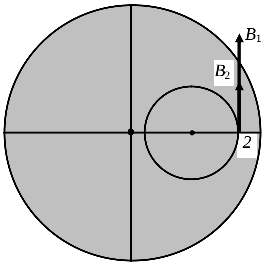


(iv) At point *2* in the figure:

$B_1 \cdot 2\pi(l+r) = \mu_0 j\pi(l+r)^2$

$B_2 \cdot 2\pi r = \mu_0 j\pi r^2$

$B = B_1 - B_2$.



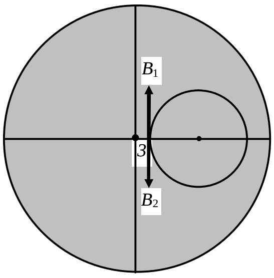
(v) At point *3* in the figure:

$B_1 \cdot 2\pi(l-r) = \mu_0 j\pi(l-r)^2$

$B_2 \cdot 2\pi r = \mu_0 j\pi r^2$

$\vec{B} = \vec{B}_1 - \vec{B}_2$

Here, $B = B_1 + B_2$ (Algebraically)

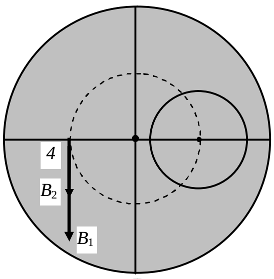


(vi) At point *4* in the figure:

$B_1 \cdot 2\pi l = \mu_0 j\pi l^2$

$B_2 \cdot 2\pi(2l) = \mu_0 j\pi r^2$

$B = B_1 - B_2$

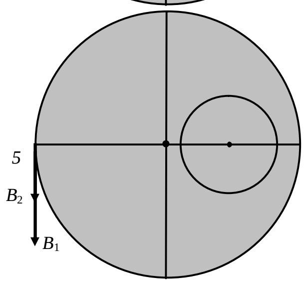


(vii) At point *5* in the figure:

$B_1 \cdot 2\pi R = \mu_0 j\pi R^2$

$B_2 \cdot 2\pi(R+l) = \mu_0 j\pi r^2$

$B = B_1 - B_2$



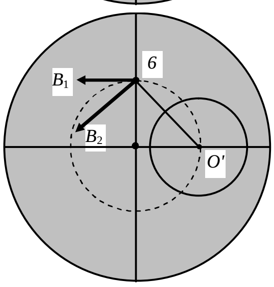
(viii) At point *6* in the figure:

$B_1 \cdot 2\pi l = \mu_0 j \cdot \pi l^2$

$B_2 \cdot 2\pi(l\sqrt{2}) = \mu_0 j\pi r^2$

$\vec{B} = B_1 - \vec{B}_2$

Vector $\vec{B}_2$ is perpendicular to the line $O'6$.



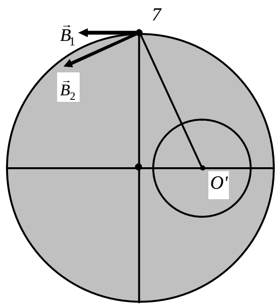
(ix) At point *7* in the figure:

$B_1 \cdot 2\pi R = \mu_0 j\pi R^2$

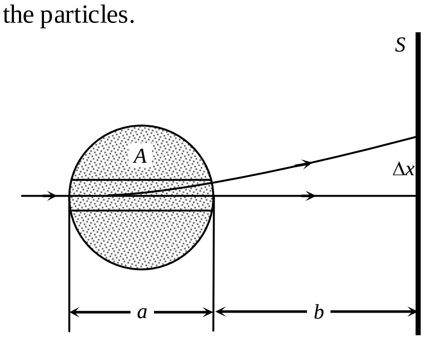
$B_2 \cdot 2\pi\sqrt{R^2 + l^2} = \mu_0 j\pi r^2$

$\vec{B} = \vec{B}_1 - \vec{B}_2$

Vector $\vec{B}_2$ is perpendicular to line $O'7$.



4. A beam of non-relativistic charged particles moves without deviation through the region of space *A*, see the figure, where there are transverse mutually perpendicular electric and magnetic fields with strength *E* and induction *B*. When the magnetic field is switched off, the trace of the beam on the screen *S* shifts by Δx . Knowing the distances *a* and *b*, find the specific charge *q*/*m* of the particles.

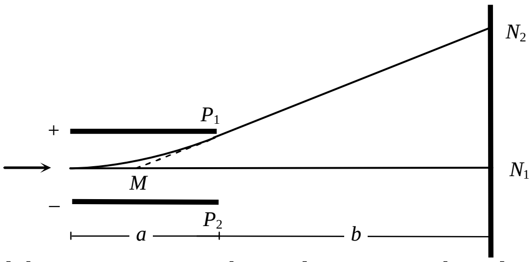


Solution: This problem is essentially the experimental setup devised by J J Thomson for determining the specific charge (*e*/*m*) of an electron, a fundamental particle whose discovery he is credited for. First, we shall understand the basic physics involved in the experiment and computation.

Suppose that a beam consisting of particles of charge *q* and mass *m* moves through a region of space where there are transverse mutually perpendicular electric and magnetic fields with strength *E* and induction *B*. If the electric and magnetic fields are so adjusted that their effects cancel, and the beam passes through the region without deviation then the electric force on the particles must be equal in magnitude and opposite in direction to the magnetic force. The electric force is given by qE , and the magnetic force by qvB , where v is the velocity of the particles. Then,

$$qE = qvB,$$

or $$v = \frac{E}{B},$$



and this experiment can be used to measure the velocity of the particles.

Next, the magnetic field is removed and the deflection of the particle caused by the electrostatic field is measured as shown in the figure. The distance N_1N_2 is the deflection caused by the electrostatic field as observed on a screen; P_1 and P_2 are the parallel plates, and a is the length of the plates. As the particles traverse the field, they are subjected to a (vertical) acceleration in a direction parallel to that of the field during the time interval $\frac{a}{v}$, so that the velocity produced in the direction of the electric field is $\left(\frac{qE}{m}\right)\left(\frac{a}{v}\right)$. On leaving the field, the velocity of the particles has the components v in the horizontal direction and $\left(\frac{qE}{m}\right)\left(\frac{a}{v}\right)=\frac{qEa}{mv}$ in the vertical direction. Hence, if N_2 is the deflected position of the beam, $\frac{N_1N_2}{MN_1}$ is the ratio of the vertical and horizontal velocities (how?):

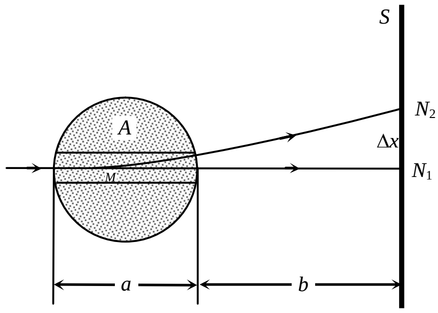
$$\frac{N_1N_2}{MN_1}=\frac{\left(\frac{qEa}{mv}\right)}{v}$$

or $$\frac{q}{m}=\frac{N_1N_2}{MN_1}\frac{v^2}{Ea}.$$

In equation, the quantities a, MN_1, and E are known from the experimental setup; the deflection N_1N_2 is measured, and v is known from the experiment in which the effects of the electric and magnetic fields cancelled each other.

Now we come to the problem. In the diagram given below

$$N_1N_2=\Delta x,\ MN_1=\frac{a}{2}+b\ .$$



Therefore, specific charge

$$\frac{q}{m}=\frac{N_1N_2}{MN_1}\frac{v^2}{Ea}=\frac{\Delta x\left(\frac{E}{B}\right)^2}{\left(\frac{a}{2}+b\right)Ea}$$

$$=\frac{2E\Delta x}{a(a+2b)B^2}.$$

You should not have any difficulty in appreciating the approximations applied in arriving at the above equation. Strictly speaking, the horizontal distance the beam covers outside the electric field is from the right edge of the upper plate to the screen, which is equal to b. Instead, we have taken this horizontal distance as $MN_1=\frac{a}{2}+b$. Why is it so? In the experimental setup, the distance of the screen from the plates is very large as compared to the length a of the plates. Therefore, $\frac{a}{2}+b\approx b$. But, even then, we could have taken the horizontal distance as b, instead of taking it as $\frac{a}{2}+b$. What made Thomson do so?

5. A ray of light traveling in air is incident at grazing angle (incident angle = 90°) on a long rectangular slab of a transparent medium of thickness t = 1.0 m (see the figure). The point of incidence is the origin $A(0, 0)$. The medium has a variable index of refraction $n(y)$ given by

$$n(y) = [ky^{3/2} + 1]^{1/2},$$

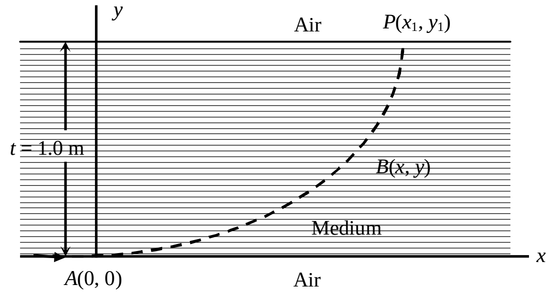
where k = 1.0 (metre)$^{-3/2}$.

The refractive index of air is 1.0.

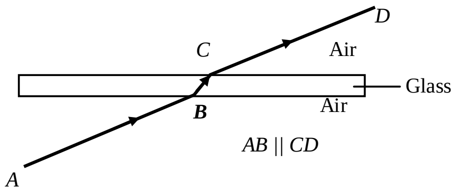
(a) Obtain a relation between the slope of the trajectory of the ray at a point $B(x, y)$ in the medium and the incident angle at that point.

(b) Obtain an equation for the trajectory $y(x)$ of the ray in the medium.

(c) Determine the coordinates (x_1, y_1) of the point P, where the ray intersects the upper surface of the slab-air boundary.



Solution: When a ray of light falls on a thin glass sheet, the emergent ray is parallel to the incident ray. This proposition can be easily demonstrated. Let us use this fact in solving this problem.



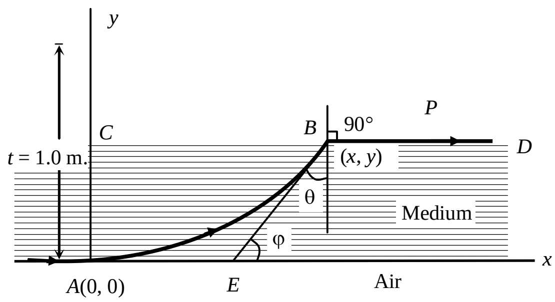
Consider the glass slab as consisting of a large number of sheets of infinitesimal thickness as shown in the figure below. These sheets have varying refractive indices. Let us truncate the slab at point B.

If the given slab is truncated at B, the emergent ray from point B will be parallel to the incident ray at point A. Why?

Let the ray after traveling through the medium fall on surface CBD at $\angle\theta$. Let EB be the tangent to the path of the ray at B. If the tangent makes an angle $\angle\varphi$ with the x-axis, slope at B equals $\tan\varphi$

(a) $$S = \frac{dy}{dx} = \tan\varphi = \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \ldots(1)$$

or $S = \cot\theta$.



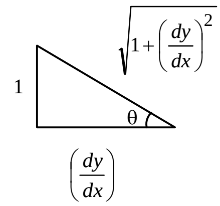
(b) Applying Snell's law at interface CBD, (with the slab truncated),

$$n \sin\theta = 1 \times \sin 90°$$

or $$\sin\theta = \frac{1}{n}. \qquad \ldots(2)$$

From equation (1), $\dfrac{dy}{dx} = \cot\theta$, which gives

$$\sin\theta = \frac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}. \qquad \ldots(3)$$



Now from equation (2) and (3),

$$\frac{1}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = \frac{1}{n} = \frac{1}{(ky^{3/2}+1)^{1/2}}$$

or $$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = (ky^{3/2}+1)^{1/2}$$

or $$1+\left(\frac{dy}{dx}\right)^2 = ky^{3/2} + 1$$

or $$\left(\frac{dy}{dx}\right)^2 = ky^{3/2}$$

or $$\frac{dy}{dx} = k^{1/2}\, y^{3/4}$$

or $$y^{-3/4}dy = k^{1/2}dx \qquad \ldots(4)$$

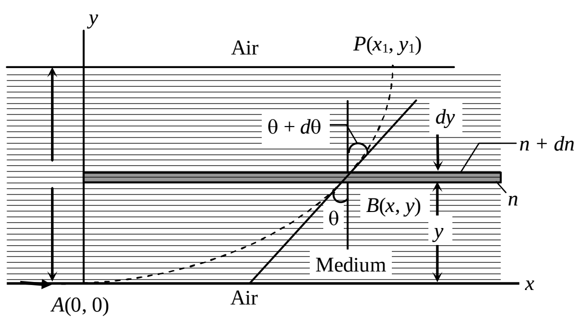
Integrating both sides of equation (4) with limits $x = 0$ to $x = x$ and $y = 0$ to $y = y$,

$$\int_0^y y^{-3/4}dy = \int_0^x k^{1/2}dx$$

or $$\frac{y^{1/4}}{1/4} = k^{1/2}x$$

or $$y = \frac{k^2x^4}{256}.$$ (Numerically $k = 1$).

Second Solution: Consider a strip of infinitesimal thickness dy at $y = y$. Let the refractive index of the medium at point $y = y$ be n. Let the refractive index change by dn over dy.



Applying Snell's law at the upper face of the strip.

$$n \sin\theta = (n + dn) \sin(\theta + d\theta)$$

or $$n \sin\theta = (n + dn) (\sin\theta \cos d\theta + \cos\theta \sin d\theta)$$

or $$n \sin\theta = (n + dn) (\sin\theta + d\theta \cos\theta)$$

or $$n \sin\theta = n \sin\theta + n\, d\theta \cos\theta + dn \sin\theta + dn\, d\theta \cos\theta$$

or $\quad dn \sin\theta = -n \cos\theta \, d\theta \qquad \begin{bmatrix} \cos d\theta = 1 \\ \sin d\theta = d\theta \\ dn \, d\theta \cos\theta = 0 \end{bmatrix}$

or $\quad \dfrac{dn}{n} = -\dfrac{\cos\theta}{\sin\theta} d\theta.$

Integrating both sides of this equation from A to B, $n = 1$ to $n = n$ and $\theta = \pi/2$ to $\theta = \theta$,

$$\int_1^n \frac{dn}{n} = -\int_{\pi/2}^{\theta} \frac{\cos\theta}{\sin\theta} d\theta$$

or $\quad \left| \ln n \right|_1^n = -\left| \ln \sin\theta \right|_{\pi/2}^{\theta}$

or $\quad \ln n = -\ln \sin\theta$

or $\quad \ln n = \ln \left(\dfrac{1}{\sin\theta} \right)$

or $\quad n = \dfrac{1}{\sin\theta}$.

Now proceed as in the previous solution.

Notice how smoothly and quickly we found n in the previous method.

(c) Equation of the trajectory is $y = \dfrac{k^2 x^4}{256}$.

At point P, $y = 1$, $x = 4$.

Therefore, $(x_1, y_1) \equiv (4, 1)$.