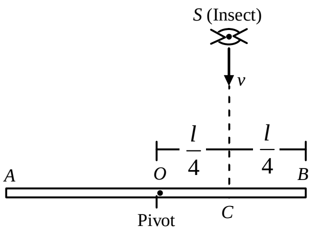
Rigorous Analysis…

SP Neelam, B Tech (IT BHU), M Tech (IIT Roorkee)

1. A homogeneous rod *AB* of length $l = 1.8$ m and mass *m* is pivoted at the centre *O* in such a way that it can rotate freely in the vertical plane, see the figure. The rod is initially in the horizontal position. An insect *S* of the same mass *m* falls vertically with speed *v* on the point *C*, mid way between the points *O* and *B*.
Immediately after falling, the insect moves towards the end *B* such that the rod rotates with a constant angular velocity ω.
(a) Determine the angular velocity ω in terms of *v* and *l*.
(b) If the insect reaches the end *B* when the rod has turned through an angle of 90°, determine *v*.
(Take $g = 10$ m/s^2.) (IIT-JEE 1992, 8 Marks)

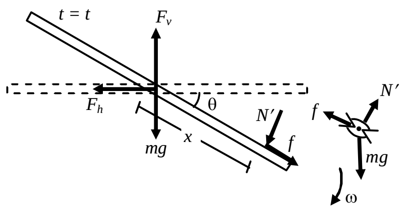


Solution : The mass of the *insect* is equal to the mass of a 1.8-m long *rod*. A very large insect indeed! Large insects have masses 50-70 grams. But their lengths vary from 10 to 15 centimeters. Here, the insect has to be treated as a particle. The clauses '…falls vertically with speed *v* on the point *C*' and '…reaches the end *B*' will assume meaning only in that case.

Also, the motion of the 'rod-insect' system, or only rod or only insect for that matter, must be governed by the laws of Particle Dynamics. The forces on the system, or on its parts, are the usual forces we study - the weight, the normal force, the frictional force, etc. However, a rigorous analysis shows that unknown forces are present here. The insect makes special efforts. The motion of the insect is in no way related to the motion of the rod except that the insect somehow remains in contact with the rod and moves towards its end *B*.

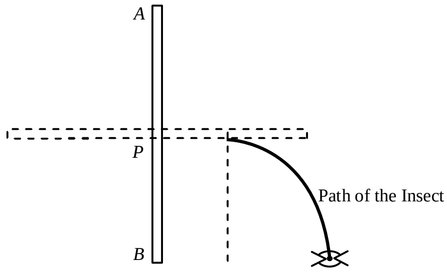
First let us apply the laws of Particle Dynamics.

After falling on the rod the insect moves towards the end *B*, *such that* the rod rotates with a constant angular velocity. This implies that the angular acceleration of the rod about the pivot is zero. And this will happen if the torque on the rod about the pivot is zero.
Consider the forces on the rod when it rotates about its pivot.



For the torque on the rod about the pivot *O* to be zero, the normal force N' on the rod by the insect must be equal to zero. And in that case, generally speaking, the frictional force f will also be equal to zero.

Now look at the forces acting on the insect. The normal force $N' = 0$, the frictional force $f = 0$. So the only force of the insect is its weight mg. The insect starts moving towards the end B of the rod immediately after falling on it. Under these conditions the path followed by the insect must be somewhat as shown in the figure.



If this is so, how can the insect reach end B when the rod has turned through 90° angle? So if the rod were to rotate with a constant angular velocity the insect would be moving as a projectile under gravity, taking the parabolic path as shown in the figure!

But there can be a frictional force on the insect, and its reaction on the rod, without the normal force N' shown in the figure above. The insect may press the rod on its sides, and thereby generate the required frictional force. This is certainly possible. But then the insect must somehow balance the component of its weight normal to the rod. There must not be any normal force on the rod by the insect.

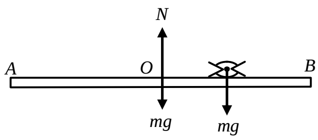
Also, for the insect, the force equation in the radial direction is $f - mg\sin\theta = m(\omega^2 x - a_{\text{rel}})$.

Here, a_{rel} is the acceleration of the insect relative to the rod. From this equation it can be seen that the force f varies in a very complicated way during the motion of the system.

Hence, the insect makes special efforts in generating unknown forces that cannot be accounted for.

Let us ignore all these details, treat the insect as a particle, ignore the torque of the unknown force about the pivot that the insect somehow generates if any, and solve the problem as given below.

The forces acting on the 'rod-insect' system during the impact is shown in the figure. The normal force on the rod by the pivot during the impact is an impulsive force, $N >> 2\ mg$. What all information do you need to calculate the value of the normal force N during the impact?



Here $\Sigma\vec{F}_{ext} \neq 0$, (convince yourself), so linear momentum of the 'rod-insect' system changes.

From equation $\vec{\tau} = d\vec{L}/dt$, you get $\vec{L}$ = constant if $\vec{\tau} = 0$. The normal force N on the rod by the pivot is a force not known to you. Can you locate a point about which the torque acting on the system is zero unless you know all the forces acting on the system?

Also, clearly, $\tau_{pivot} = mg.\frac{l}{4} \neq 0$. Here $\left(mg \cdot \frac{l}{4}\right)\Delta t = 0$ as Δt is negligibly small. So under 'impulse approximation' the angular momentum of the 'rod-insect' system about the pivot can be conserved. *(Why ?)*

$$\text{or} \quad mv\frac{l}{4} = \left(\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right)\omega$$

$$\text{or} \quad mv\frac{l}{4} = \left(\frac{1}{12} + \frac{1}{16}\right)ml^2\omega$$

$$\text{or} \quad \omega = \frac{12}{7}\frac{v}{l}.$$

The insect moves towards end B such that the rod rotates with constant angular velocity. There will always be a torque on the system about the pivot. This torque does not result in an angular acceleration otherwise ω will change. The equation $\vec{\tau} = I\vec{\alpha}$ holds for a system where the moment of inertia I about the axis of rotation is constant. Here the moment of inertia of the 'rod-insect' system changes with time.

But even if the moment of inertia about the axis of rotation changes, we can apply the equation $\vec{\tau} = \frac{d\vec{L}}{dt}$.

Using this equation we get

$$mgx\cos\theta = \frac{d}{dt}\left\{\left(\frac{ml^2}{12} + mx^2\right)\omega\right\}$$

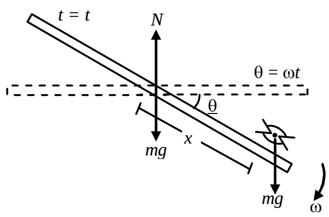
$$\text{or} \quad mgx\cos\theta = 2mx\frac{dx}{dt}\omega$$

$$\text{or} \quad g\cos\theta = 2\omega\frac{dx}{dt}. \qquad \ldots(1)$$

This is the differential equation of motion of the system. Substituting $\theta = \omega t$ in this equation,

$$g\cos\omega t = 2\omega\frac{dx}{dt}$$

$$\text{or} \quad g\cos\omega t\ dt = 2\omega dx. \qquad \ldots(2)$$



The rod turns through $\pi/2$ in time $(\pi/2)/\omega$ and the position of the insect changes from $x = l/4$ to $x = l/2$.

Integrating both the sides of equation (2) with these limits

$$\int_0^{\pi/2\omega} g\cos\omega t\ dt = \int_{l/4}^{l/2} 2\omega\ dx$$

$$\text{or} \quad \frac{g}{\omega}|\sin\omega t|_0^{\pi/2\omega} = 2\omega|x|_{l/4}^{l/2}$$

$$\text{or} \quad \frac{g}{\omega}\left(\sin\omega.\frac{\pi}{2\omega} - \sin 0\right) = 2\omega\left(\frac{\ell}{2} - \frac{\ell}{4}\right)$$

$$\text{or} \quad \frac{g}{\omega} = 2\omega\frac{l}{4}$$

$$\text{or} \quad \omega^2 = \frac{2g}{l}$$

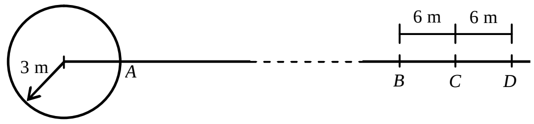
$$\text{or} \quad \left(\frac{12}{7}\frac{v}{l}\right)^2 = \frac{2g}{l} \qquad \left(\omega = \frac{12}{7}\frac{v}{l}\right)$$

$$\text{or} \quad v = \frac{7}{12}\sqrt{2gl} = \frac{7}{12}\sqrt{2\times 10\times 1.8} = 3.5 \text{ m/s}.$$

---

2. A source of sound is moving along a circular orbit of radius 3 m with an angular velocity of 10 rad/s. A sound detector located far away from the source is executing simple harmonic motion along the line BD (see the figure) with an amplitude $BC = CD = 6$ meters. The frequency of oscillation of the detector is $5/\pi$ per second. The source is at point A when the detector is at point B. If the source emits a continuous sound wave of frequency 340 Hz find the maximum and the minimum frequencies recorded by the detector. Take the velocity of sound in air as 330 m/s.

(IIT-JEE 1990, 7 Marks)

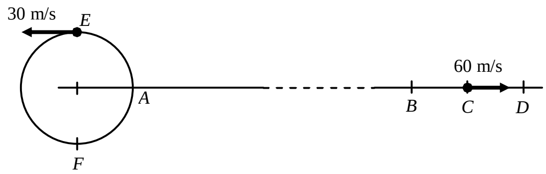


Solution: The answers, wherever the problem has been solved or given, are υ_{max} = 442 Hz and υ_{min} = 225 Hz. These answers are not correct. The logic on the basis of which they arrive at these answers has flaws. In fact, this problem cannot be solved because a crucial fact has been neglected while framing the problem.

First, let us see how these answers have been calculated.

For source	For observer
ω = 10 rad/s	$v = 5/\pi$ per second.
Radius of the circular orbit r = 3 m.	Angular frequency ω = $2\pi\upsilon$ = 10 rad/s.
Speed $v = r\omega$ = 30 m/s.	Amplitude of SHM = 6m
Time period $T = \frac{2\pi}{\omega} = \frac{\pi}{5}$ s.	Maximum speed, at the mean position $a\omega$ = 60 m/s.

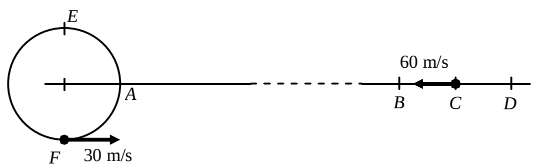
Let at time t = 0, there is no loss of generality in assuming so, the source be at point A.



The source will be at E (with velocity 30 m/s towards left) at the moments

$$t = \frac{\pi}{20}, \frac{\pi}{20}+\frac{\pi}{5}, \frac{\pi}{20}+\frac{2\pi}{5}, \frac{\pi}{20}+\frac{3\pi}{5}, \frac{\pi}{20}+\frac{4\pi}{5}, \ldots$$

And at these moments the observer will be at C with velocity 60 m/s towards right.

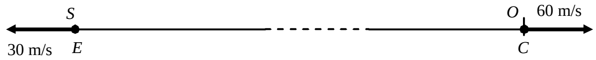


The source will be at F (velocity 30 m/s towards right) at the moments

$$t = \frac{3\pi}{20}, \frac{3\pi}{20}+\frac{\pi}{5}, \frac{3\pi}{20}+\frac{2\pi}{5}, \frac{3\pi}{20}+\frac{3\pi}{5}, \ldots$$

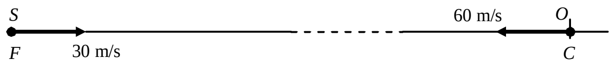
And at these moments the observer will be at C with velocity 60 m/s towards left.

As the source is far away from the observer, the velocities of the source and the observer can be taken along the same line, and the maximum and the minimum frequencies recorded can be calculated as under : For the moments when the source is at E,



$$\upsilon_{\min} = \upsilon_0 \frac{v - v_o}{v + v_s} = 340 \times \frac{330-60}{330+30} = 225 \text{ Hz.}$$

And when the source is at F,



$$\upsilon_{\max} = \upsilon_0 \frac{v + v_o}{v - v_s} = 340 \frac{330+60}{330-30} = 442 \text{ Hz.}$$

Now, we will check whether the given information also meets the requirements of an essential condition.

Will the note emitted by the source while at point E (moving towards left with 30 m/s) necessarily be received by the observer when it is at C (moving towards right with velocity 60 m/s)? After all the distance between the source and the observer is long enough… 'the detector located far away from the source'…; the wavelengths emitted by source when it is at point E will reach the observer only after some time, it will never reach the detector instantly. The time taken by the sound to travel from the source to the detector can not be neglected. And this time need not be an integral multiple of $\pi/5$ s!

The notes emitted by the source when it is at point E $\left(\text{at } t = \frac{\pi}{20}, \frac{\pi}{20}+\frac{\pi}{5}, \frac{\pi}{20}+\frac{2\pi}{5}, \ldots\right)$ will certainly be detected by the observer, but the position and velocity of the detector at the moment it detects these notes can not be ascertained; detector can be anywhere on the line BD, and its velocity can have any value from 60 m/s towards left to 60 m/s towards right (-60 m/s $\le v_0 \le +60$ m/s).

Where did the examiner falter?

He combined two problems in one.

(1) A source moves along a small circle and a detector located far away on the line through the centre of the circle and in the plane of the circle. (2) A detector performs simple harmonic motion on a line and a source located far away on that line. These two problems can easily be solved. The moment of emission of a note and the moment of detection are not important here. But in the given problem they are. And the examiner did not leave the choice of selecting these moments of time to students. He clearly stated that **the source was at point *A* when the detector was at point B. Probably that is where he faltered.**

3. A cylindrical block of length 0.4 m and area of cross-section 0.04 m^2 is placed coaxially on a thin metal disc of mass 0.4 kg and of same cross-section. The upper face of the cylinder is maintained at a constant temperature of 400 K and the initial temperature of the disc is 300 K. If the thermal conductivity of the material of the cylinder is 10 Watt/m-K and the specific heat of the material of the disc is 600 J/kg-K, how long will it take for the temperature of the disc to increase to 350 K? Assume, for the purposes of calculation, the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper face of the cylinder. (IIT-JEE 1992, 8 Marks)

Solution: Here, apart from numerical values, you have been given two conditions.

(i) The thermal conductivity of the disc is very high.

(ii) The system is thermally insulated except for the upper face of the cylinder.

From the first condition you infer that the heating of the disc is uniform, there is no temperature gradient across the disc; the use of the phrase 'the temperature of the disc' is justified. The rate at which heat enters the disc, is the rate of the temperature rise of the disc given by $\frac{dq}{dt} = m\,c\,\frac{dT}{dt}$, where m and c are the mass of the disc and the specific heat of the disc material respectively.

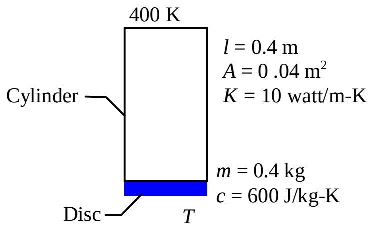
From the second condition you infer that there is no heat loss to the surrounding. All the heat that enters the top of the cylinder goes to the disc from its bottom.

But how can you account for the heat absorbed by the cylinder? After all, as the heat flow takes place, and as the temperature of the disc increases, the temperature at different cross-sections of the cylinder will also increase. Don't you think some heat would be absorbed in raising the temperature at different cross-sections of the cylinder? Of course, it would be. You can use the heat flow equation $\frac{dq}{dt} = \frac{kA(\Delta T)}{l}$ if at all the moments of time the heat flow is under steady state condition, the heat entering the upper face of the cylinder is equal to the heat leaving its bottom face, no heat being absorbed by the cylinder. Then how does the temperature at the different cross-sections of the cylinder rise?

So with the given conditions the problem can not be solved.

You can solve the problem only if you assume that the **specific heat of the material of cylinder is zero** (or negligible). That is, the temperature changes at different cross-sections of the cylinder do not involve any absorption of heat. With this condition, the problem can be solved as follows.

Let the temperature of the disc at time t be T. Let the increment of the temperature of the disc in time dt be dT.



Rate of heat flow through the cylinder = Rate of heat flow to the disc.

or $$\frac{KA(400-T)}{l} = mc\frac{dT}{dt}$$

or $$\frac{KA}{l}dt = mc\frac{dT}{400-T}$$

Integrating both sides with suitable limits,

or $$\frac{KA}{l}t = -mc \left| \ln(400-T) \right|_{300}^{350}$$

$$= -mc\,(\ln 50 - \ln 100)$$

or $$\frac{KA}{l}t = mc\ln 2$$

or $$t = \frac{lmc}{KA}\ln 2.$$

Substituting the given numerical values,

$$t = \frac{0.4\times 0.4\times 600}{10\times 0.04}\ln 2 = 240 \ln 2 \text{ s} = 166.3 \text{ s}.$$

4. An electron and a proton are moving on straight parallel paths with same velocity. They enter a semi-infinite region of uniform magnetic field perpendicular to the velocity. Which of the following statement(s) is/are true?

(A) They will never come out of the magnetic field region.

(B) They will come out traveling along parallel paths.

(C) They will come out at the same time.

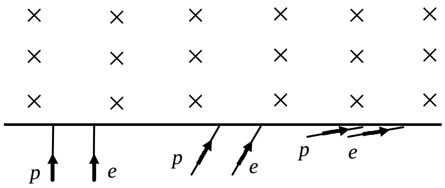
(D) They will come out at different times.

(IIT-JEE 2011, 2 Marks)

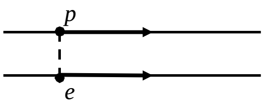
Solution: In the key given on the official IIT-JEE website the answer was **(B, C)** or **(B, D)** or **(B, C, D)**. This question has three answers! Strange isn't it? How on the earth choices (C) and (D) can be true simultaneously?

This question is very confusing. It can be easily shown that choice (B) is correct. Choice (C) may be correct, may not be correct. Similarly choice (D) may be correct, may not be correct. On the basis of the information given in the question one cannot decide it.

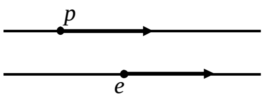
The particles enter a semi-infinite region of uniform magnetic field perpendicular to the velocity. We are given the angle the velocity vector makes with the magnetic field, but not with the boundary of the region. The following figure shows three cases out of infinite ways the particles can enter the magnetic field.



Note that the phrase "an electron and proton moving on straight parallel paths with same velocity." does not necessarily mean that the two particles are side by side as shown in the figure below.



The two particles can be moving with same velocity on parallel paths as shown in the figure below, and enter the magnetic field at the same instant.



We know that a proton is 1836 times (approximately) heavier than an electron. Once they enter the magnetic field, their angular velocities will be different due to the difference in their masses.

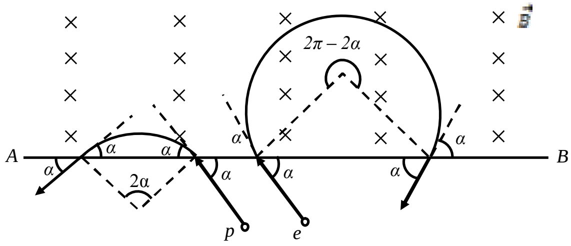
Angular velocity of proton $\omega_p = \dfrac{eB}{m_p}$, and

angular velocity of electron $\omega_e = \dfrac{eB}{m_e}$,

where B is the magnetic field, e is the magnitude of the charge on proton and electron, m_e mass of electron and m_p mass of proton.

Clearly, $\omega_e >> \omega_p$.

From this information can we really conclude that the time taken by both the particles to come out of the magnetic field will be different? Suppose the path taken by electron is longer than that taken by proton. Then there is a possibility that they come out at the same time. Consider the following situation.



Let the proton and electron enter in parallel paths at an angle α as in the figure above. AB is the boundary separating the magnetic field region from non-magnetic field region. The line AB extends to infinity.

The path of proton subtends angle 2α at the centre of the circle, while the path of electron subtends (2π -2α) at the centre of the circle. Do the required geometrical calculations yourself.

Hence time taken by the electron is

$t_e = \dfrac{(2\pi - 2\alpha)m_e}{eB}$, and that by the proton is

$t_p = \dfrac{2\alpha m_p}{eB}$.

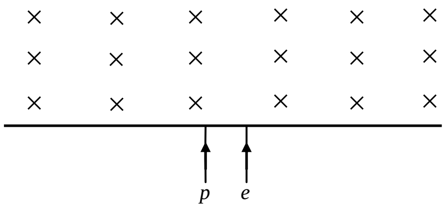
Now if $t_e = t_p$, then

$(\pi - \alpha)\, m_e = \alpha\, m_p$

or $\qquad \alpha = \dfrac{\pi m_e}{m_e + m_p} = \dfrac{\pi}{1837}$.

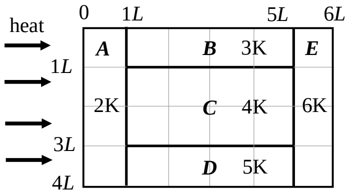
This is a very small angle. But conceptually this is a possibility. And if the particles enter the magnetic field at this angle then the time taken by them to come out of it will be equal. In that case choice (C) will be correct. In other situations choice (D) will be correct.

It seems that the examiner assumed that the velocities of electron and proton were same, they were perpendicular to the magnetic field and also **perpendicular to the boundary**, as shown in the figure below.



But it was not written in the paper clearly. Paper did not have any diagram also.

5. A composite block is made of slabs A, B, C, D and E of different thermal conductivities (given in terms of a constant K) and sizes (given in terms of length, L) as shown in the figure. All slabs are of same width. Heat 'Q' flows only from left to right through the blocks. Then in steady state



(A) heat flow through A and E slabs are same.

(B) heat flow through slab E is maximum.

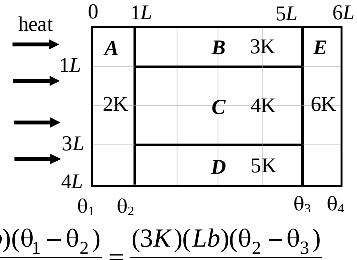
(C) temperature difference across slab E is smallest.

(D) heat flow through C = heat flow through B + heat flow through D. (IIT-JEE 2011, 3 Marks)

Solution: As per the official IIT-JEE site the answer of this question is (**A**), (**C**), (**D**). But a rigorous analysis shows that the choice (**C**) is not correct, because under the given conditions the behavior of slab E is pretty strange, and the phrase 'temperature difference across slab E ' does not have any meaning. It also turns out that the choice (**D**) is also incorrect.

This question has many flaws. If you look at the arrangement carefully, you can argue intuitively that the behaviors of slabs A and E are strange. Thermal conductivity of slab D is more than that of slabs C and B. Shouldn't heat rush to take the path of least thermal resistance? Shouldn't heat flow through slab A from top to bottom, and through E from bottom to top too? To visualize this, consider an unusual situation. Assume that the thermal conductivity of the slabs B and C is zero. Heat flow occurs only through slab D. What will be the pattern of heat flow through slabs A and E in that case?

Anyway, let us go by what has been given in the question. All the slabs allow heat flow only from left to right. For lateral heat flow they offer infinite thermal resistance. Assume the temperature at the interfaces as shown in the figure below. If the thickness of each slab is b, in steady state heat flow condition

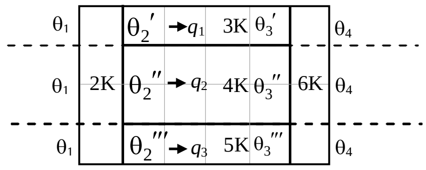


$$\frac{(2K)(4Lb)(\theta_1-\theta_2)}{L} = \frac{(3K)(Lb)(\theta_2-\theta_3)}{4L} + \frac{(4K)(2Lb)(\theta_2-\theta_3)}{4L} + \frac{(5K)(Lb)(\theta_2-\theta_3)}{4L} = \frac{(6K)(4Lb)(\theta_3-\theta_4)}{L}$$

$\Rightarrow 2(\theta_1-\theta_2) = (\theta_2-\theta_3) = 6(\theta_3-\theta_4)$. …(1)

If θ_1 and θ_4 are fixed, you can find θ_2 and θ_3 in terms of θ_1 and θ_4.

Now since heat flows through the slabs only from left to right, you can also write heat flow equations as



$$q_1 = \frac{(2K)(Lb)(\theta_1-\theta_2')}{L} = \frac{(3K)(Lb)(\theta_2'-\theta_3')}{4L} = \frac{(6K)(Lb)(\theta_3'-\theta_4)}{L}$$

$$\Rightarrow 2(\theta_1-\theta_2') = \frac{3}{4}(\theta_2'-\theta_3') = 6(\theta_3'-\theta_4)\,. \quad \ldots(2)$$

$$q_2 = \frac{(2K)(2Lb)(\theta_1-\theta_2'')}{L} = \frac{(4K)(2Lb)(\theta_2''-\theta_3'')}{4L} = \frac{(6K)(2Lb)(\theta_3''-\theta_4)}{L}$$

$$\Rightarrow 4(\theta_1-\theta_2'') = 2(\theta_2''-\theta_3'') = 12(\theta_3''-\theta_4)\,. \quad \ldots(3)$$

And

$$q_3 = \frac{(2K)(Lb)(\theta_1-\theta_2''')}{L} = \frac{(5K)(Lb)(\theta_2'''-\theta_3''')}{4L} = \frac{(6K)(Lb)(\theta_3'''-\theta_4)}{L}$$

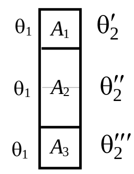
$$\Rightarrow 2(\theta_1-\theta_2''') = \frac{5}{4}(\theta_2'''-\theta_3''') = 6(\theta_3'''-\theta_4)\,. \quad \ldots(4)$$

Look at the equations (1) through (4) carefully. Without solving them you can conclude that

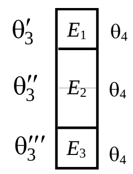
$\theta_2 \neq \theta_2' \neq \theta_2'' \neq \theta_2'''$ and $\theta_3 \neq \theta_3' \neq \theta_3'' \neq \theta_3'''$.

We can draw the following conclusions

1. Slab A is made of 3 slabs.



2. Slab E is made of 3 slabs.



3. Temperature difference across slab E has no meaning. It is $(\theta_3' - \theta_4)$ for some part, $(\theta_3'' - \theta_4)$ for some other part and $(\theta_3''' - \theta_4)$ for still other part.

So choice (C) cannot have any meaning.

4. The temperature is not same everywhere at the right end of slab A. Also the temperature is not same everywhere at the left end of slab E. Under these conditions, the formulation of equation (1) is not correct.

5. Heat flow through slabs $A_1 \to B \to E_1$

$$R_1 = \frac{L}{(2K)(Lb)} + \frac{4L}{(3K)(Lb)} + \frac{L}{(6K)(Lb)} = \frac{2}{Kb}$$

$$q_1 = \frac{\theta_1 - \theta_4}{\left(\frac{2}{Kb}\right)}.$$

Heat flow through slabs $A_2 \to C \to E_2$

$$R_2 = \frac{L}{(2K)(2Lb)} + \frac{4L}{(4K)(2Lb)} + \frac{L}{(6K)(2Lb)} = \frac{10}{12Kb}$$

$$q_2 = \frac{\theta_1 - \theta_4}{\left(\frac{10}{12Kb}\right)}.$$

Heat flow through slabs $A_3 \to D \to E_3$

$$R_3 = \frac{L}{(2K)(Lb)} + \frac{4L}{(5K)(Lb)} + \frac{L}{(6K)(Lb)} = \frac{22}{15Kb}$$

$$q_3 = \frac{\theta_1 - \theta_4}{\left(\frac{22}{15Kb}\right)}.$$

Clearly, $q_1 + q_3 \neq q_2$.

Hence choice (D) is not correct.

In the steady state, the rate of heat flow through any cross section is same $\Rightarrow$ choice (A) is correct.

We have, rate of heat flow

$$\frac{dQ}{dt} = \frac{KA(\theta_2 - \theta_1)}{l}, \qquad \ldots(1)$$

where K is thermal conductivity of the material, A is area of cross section of slab and l the length, $\theta_2 - \theta_1$ is temperature difference across the ends of slab.

Now for the slabs B, C and D the temperature difference is same.

Hence the rates of heat flow through these slabs depend on the factor $\frac{KA}{l}$.

Let us represent this factor by β.

For slab B, $\beta \equiv \frac{3K(L\times b)}{4L} = \frac{3Kb}{4}$

For slab C, $\beta \equiv \frac{4K(2L\times b)}{4L} = 2Kb$

For slab D, $\beta = \frac{5K(L\times b)}{4L} = \frac{5Kb}{4}$

Clearly, $\frac{3Kb}{4} + \frac{5Kb}{4} = 2Kb.$

$\Rightarrow$ heat flow through slab B + heat flow through slab D = heat flow through C.

The quantity $\frac{l}{KA}$ is called thermal resistance. For a given heat flow rate, the temperature difference is smallest if thermal resistance is smallest.

For slab A thermal resistance is $\frac{1}{8}\alpha$ (α is some constant).

For slabs ($B + C + D$) it is $\frac{1}{4}\alpha$.

For slab E it is $\frac{\alpha}{24}$.

For slab E the thermal resistance, hence the temperature difference across it, is the smallest.

---

6. Four point charges, each of $+q$, are rigidly fixed at the four corners of a square planar soap film of side 'a'. The surface tension of the soap film is γ. The system of charges and planar film are in equilibrium, and $a = k\left[\frac{q^2}{\gamma}\right]^{1/N}$, where '$k$' is a constant. Then N is

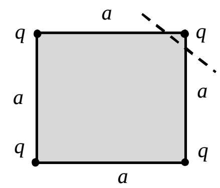
(IIT-JEE-2011, 4 Marks)

Solution: This question defies every conceivable logic.

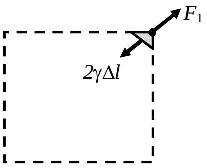
There is a fundamental flaw in framing this problem.

It has been given in the question that the point charges are rigidly fixed at the corners of a square planar soap film. *How*? How do you rigidly fix a charged particle to a liquid film.

Anyway, let us go by what examiner says and assume that somehow it has been done. The square planar soap film of side a with point charge $+q$ rigidly fixed at each of its corners is in equilibrium. See the figure below.

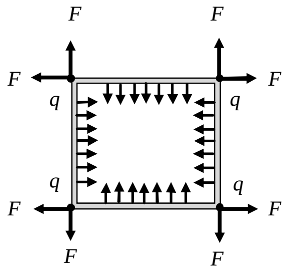


Cut the film along the dotted line shown. Draw the forces on the point charge and the small part of the soap film with it.



The electrostatic repulsion F_1 on the charge has a finite value. The surface tension force on the small part of the film is infinitesimally small. We have the liberty to make Δl as small as we please. Can this part of the system ever be in equilibrium?. How can the net force on this system be zero?

Now consider another system: the four charges and a very thin thread of the soap film. Forces on this system are shown in the figure. The columbic force on each charge has been shown in two components.

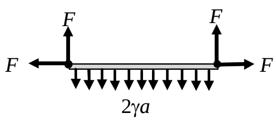


Total surface tension force on each side = $2\gamma\, a$.

One can say that if $2F = 2\gamma a$, the above system is in equilibrium.

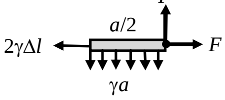
Surely, it is. Don't you see an elephant hanging from a thin cotton thread in this system, tension in the thread being equal to the weight of the elephant!

Consider another system of two point charges and a thread of the soap film.



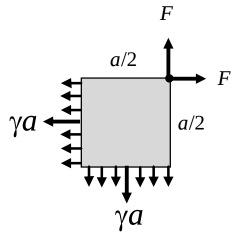
One can say that if $2F = 2\gamma a$ net force on the system is zero, it is in equilibrium. But what about the forces which act along the thread of the soap film? Will they not break it apart?

The following system is still a better choice:

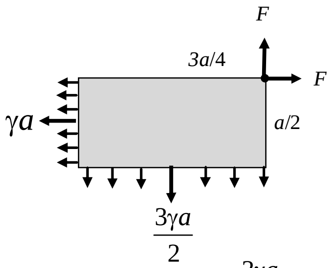


$F = \gamma a$, and $F = 2\gamma\Delta l$ too?

You may be tempted to consider a finite part of the film. Two such systems are given below.



$F = \gamma a$. Fine!

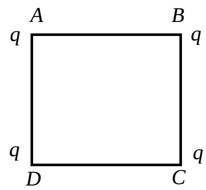


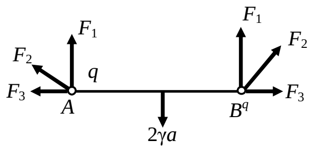
$F = \gamma a$, and $F = \frac{3\gamma a}{2}$ too?

Can you now find the mistake in the framing of the problem? Possibly yes.

Now let us solve the problem the way examiner wanted us to do it.

We take the line *AB* and equate the net vertical force on *AB* to zero.





$$F_1 = \frac{q^2}{4\pi\,\varepsilon_0\, a^2} = \frac{k'q^2}{a^2}$$

$$F_2 = \frac{q^2}{4\pi\,\varepsilon_0\,(\sqrt{2}a)^2} = \frac{k'q^2}{2a^2}$$

Hence, $2F_1 + 2F_2\cos\frac{\pi}{4} = 2\gamma a$

or $\quad \frac{2k'q^2}{a^2} + \frac{\sqrt{2}k'q^2}{2a^2} = 2\gamma a$

or $\quad a^3 = \frac{q^2 k'\left(1+\frac{1}{2\sqrt{2}}\right)}{\gamma} = \frac{k^3 q^2}{\gamma},$

where $k^3 = k'\left(1+\frac{1}{2\sqrt{2}}\right)$

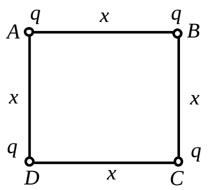
$$\Rightarrow a = k\left[\frac{q^2}{\gamma}\right]^{1/3} \Rightarrow N = 3.$$

A smart idea: If you read the problem carefully, you can see that the value of N can be found out without doing any calculations. The problem has been worded in such a way that you can arrive at the answer without calculating the coulomb force on the charges. From Coulomb's law and principle of superposition you can write that the Coulomb's force on a charge, F_c = (a constant) $\frac{q^2}{a^2}$.

Just write this constant as $2k^3$. Then $\frac{2k^3q^2}{a^2} = 2\gamma a$, which gives $a = k\left[\frac{q^2}{\gamma}\right]^{1/3}$.

I am sure that many students have thought of this and have done this problem in exactly this way in the examination hall. Many brilliant students might have wasted precious exam time in figuring out the problem. Those who are in the habit of doing things correctly may have been victims of great confusion and perplexity.

Second Solution: Let x be the side of the square as shown in the figure below.



For the equilibrium condition the electrostatic potential energy + surface energy must be a minimum. Let the equilibrium be attained at $x = a$.

Now, the total energy,

$$E = 4\cdot\frac{q^2}{4\pi\varepsilon_0 x} + 2\cdot\frac{q^2}{4\pi\varepsilon_0\sqrt{2}x} + \gamma(2x^2)$$

$$= \frac{(4+\sqrt{2})}{4\pi\varepsilon_0}\frac{q^2}{x} + 2\gamma x^2.$$

For equilibrium $\frac{d(E)}{dx} = 0$,

or $$-\frac{(4+\sqrt{2})}{4\pi\varepsilon_0}\left(\frac{q^2}{x^2}\right) + 4\gamma x = 0$$

or $$x^3 = \frac{\left(1+\frac{1}{2\sqrt{2}}\right)q^2}{4\pi\varepsilon_0\gamma} \equiv \frac{k^3q^2}{\gamma}$$

$$\Rightarrow \quad x = k\left[\frac{q^2}{\gamma}\right]^{1/3}.$$

Since the equilibrium is attained at $x = a$, $a = k\left[\frac{q^2}{\gamma}\right]^{\frac{1}{3}}$. Hence $N = 3$.

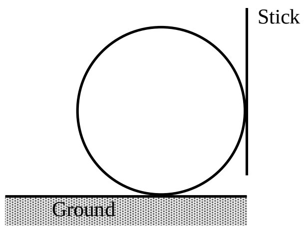
A smart idea again: You can arrive at the answer smartly without doing any calculations. Write the total mechanical energy assuming that the side of the square is x as

$$E = C\frac{q^2}{x} + 2x^2\gamma$$

Minimize E for x. Let the equilibrium be attained at $x = a$. Replace the constant term by k intelligently.

---

7. A boy is pushing a ring of mass 2 kg and radius 0.5 m with a stick as shown in the figure. The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s^2. The coefficient of friction between the ground and the ring is large enough that rolling always occurs and the coefficient of friction between the stick and the ring is (P/10). The value of P is



(IIT-JEE 2011, 4 Marks)

Solution: As per official IIT-JEE website answer of this question is ***P* = 4**. It was written explicitly in the paper that answer is a single digit integer ranging from 0 to 9. Students faced great difficulty while solving this problem.

'The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s^2'. This statement is not entirely incorrect, but it is certainly confusing. What does the examiner mean by 'the stick applies a force of 2 N on the ring'? Is it the normal force on the ring by the stick or the total force on the ring by the stick? In the first sentence of the question it is given that the boy is pushing the ring. On this basis it can be said that 2 N is the normal force acting on the ring by the stick, not the total force.

Let the coefficient of friction between the stick and the ring be μ. The forces acting on the ring are:

1. 2-N normal force by the stick.

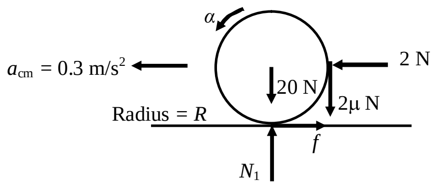
2. Frictional force of value 2μN vertically downward. Note that this frictional force is kinetic in nature. As the ring rolls on the ground, it rotates about its centre of mass in anticlockwise sense. The point of contact of the ring with stick is in motion. The frictional force is opposite to the velocity of point of contact.

3. Weight of the ring 2×10 N vertically downward acting at the centre of gravity.

4. Normal force N_1 at the point of contact by the ground.

5. Frictional force f at the point of contact with the ground. This friction is static in nature. Since the ring always rolls on the ground, the frictional force on the ring by the ground is static friction less than its limiting value. The direction of this friction is towards the right. Why? The tendency of motion of the point of contact is towards left. This is due to the action of 2-N force.

The free body diagram of the ring is as shown in the figure below.



The force and torque equations for the ring are

$2 - f = 2 \times a_{cm} = 2 \times 0.3$ …(1)

$N_1 - 20 - 2\mu = 0$ …(2)

$f \times 0.5 - 2\mu \times 0.5 = 2 \times (0.5)^2 \times \alpha$ …(3)

The constraint equation is

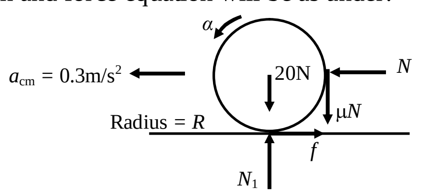
$a_{cm} = R\,\alpha$ or $0.3 = 0.5 \times \alpha$ …(4)

From these equations

$\mu = 0.4$

So, $\frac{P}{10} = 4.0$ or $P = 4$.

But one can also read the problem in a different way. One can assume that 2-N force is the total force acting on the ring by the stick. In that situation the free body diagram and force equation will be as under.



Radius of the ring $R = 0.5$ m.

Acceleration of center of mass, $a_{cm} = 0.3$ m/s^2.

Since the ring rolls on the ground, its angular acceleration $\alpha = \frac{\alpha_{cm}}{R} = \frac{0.3}{0.5} = \frac{3}{5}$ radian/s^2.

Now the force and torque equations are

$N - f = m\, a_{cm} = 2 \times 0.3 = 0.6$ ...(1)

$f\, R - \mu NR = I_{cm}\,\alpha$

or $f \times 0.5 - \mu N \times 0.5 = 2 \times (0.5)^2 \times \frac{3}{5}$

or $f - \mu N = 0.6$ …(2)

From equations (1) and (2),

$N = \frac{1.2}{1-\mu}$.

Now if the total force on the ring by the stick is 2 N,

$\sqrt{N^2 + (\mu N)^2} = 2$

or $\sqrt{\left(\frac{1.2}{1-\mu}\right)^2 + \left(\frac{1.2\mu}{1-\mu}\right)^2} = 2$

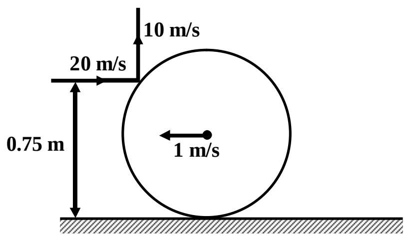
or $\mu = 0.361, 2.763$.

$\Rightarrow$ P = 3.61, 27.63, which is not a SINGLE-DIGIT INTEGER.

Many students wasted lots of precious exam time on this problem. They took the 2 N force as the total force on the ring by the stick.

---

8. A thin ring of mass 2 kg and radius 0.5 m is rolling without slipping on a horizontal plane with velocity 1 m/s. A small ball of mass 0.1 kg, moving with velocity 20 m/s in the opposite direction, hits the ring at a height of 0.75 m and goes vertically up with velocity 10 m/s. Immediately after the collision,



(A) the ring has pure rotation about its stationary CM.

(B) the ring comes to a complete stop.

(C) friction between the ring and the ground is to the left.

(D) there is no friction between the ring and the ground.

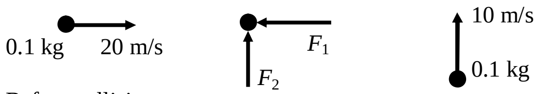
(IIT-JEE 2011 3 Marks)

Solution: The answer is **A** or **A,C** as per official IIT-JEE key. The question created a lot of confusion and stir in the country. The examiner took great care in choosing the values of masses, velocities and lengths. He faltered in phrasing choice (C). Probably, to make up for his misdoings, IIT-JEE organizing body came with two answers: A or AC. But a rigorous analysis shows that even choice (A) cannot be correct.

The examiner did not consider the frictional force on the ring by the ground during the collision. The values of masses and velocities and the answer given on IIT-JEE site corroborate this fact. While solving this problem if you do not consider the frictional force on the ring by the ground then you can show that just after the collision the velocity of centre of mass of the ring is zero; but the ring has an angular velocity about the centre of mass and the frictional force on the ring by the ground is toward left. But the examiner somewhat mis-worded it: *the friction between the ring and ground is to the left.*

In realty, a frictional force, impulsive in nature, that is, of a relatively large value, acts on the ring during the collision. Whether this kind of frictional force is to be considered or not created a great deal of confusion.

To begin with, we consider the examiner's point of view. First analyze the motion of the ball. Suppose the forces acting on the ball are as shown in the figure.



Before collision Forces on ball during collision After collision

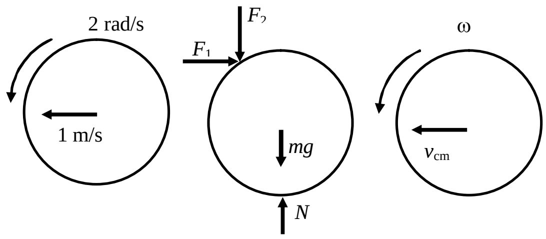
If the time of collision is Δt,

$F_1 \cdot \Delta t = 0.1 \times 20 = 2$ N·s …(1)

$F_2 \cdot \Delta t = 0.1 \times 10 = 1$ N·s …(2)

(Please take care of signs and directions.)

Next consider the ring. Ignoring the frictional force during the collision, the forces on the ring are as shown in the figure:



Before collision During collision After collision

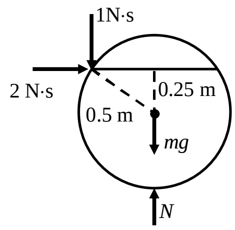
Linear impulse = change in linear momentum gives

$-F_1\,\Delta t = 2\,v_{cm} - 2 \times 1$

or $-2 = 2\,v_{cm} - 2$ (From (1))

or $v_{cm} = 0$.

And angular impulse about CM = change in angular momentum about CM gives,

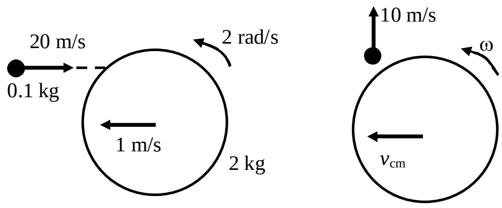


$$1 \times 0.5 \times \frac{\sqrt{3}}{2} - 2 \times 0.25 = 2 \times (0.5)^2 (\omega - 2)$$

or $\omega = 1.866$ rad/s anticlockwise.

That is, immediately after collision, $v_{cm} = 0$ and $\omega = 1.866$ rad/s anticlockwise. The ring rotates about stationary centre of mass. Velocity of the point of contact with ground is to the right. The friction on the ring by the ground is to the left.

You can also arrive at these answers by conserving linear momentum, and angular momentum of the ring ball system relative to the ground.



Just before collision Just after collision

Conservation of linear momentum

$-0.1 \times 20 + 2 \times 1 = 2 \times v_{cm} \Rightarrow v_{cm} = 0.$

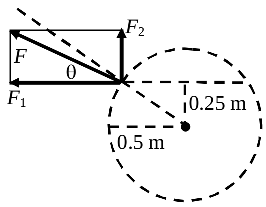
Conservation of angular momentum relative to the ground

$$\underbrace{2\times1\times0.5}_{M\,v_{cm}\,r} + \underbrace{2\times(0.5)^2\times2}_{I_{cm}\,\omega_1} - \underbrace{0.1\times20\times0.75}_{mv_i r_\perp}$$

$$= \underbrace{2\times(0.5)^2\,\omega}_{I_{cm}\,\omega_2} - \underbrace{0.1\times10\times0.5\times\frac{\sqrt{3}}{2}}_{m\,v_f\,r_\perp}$$

$$\Rightarrow \omega = 1 + \frac{\sqrt{3}}{2} = 1.866 \text{ rad/s.}$$

Now let us check whether the force on the ball by the ring is normal to the ring or not. The situation is shown in the figure below.



From the above figure,

$\tan\theta = \dfrac{F_2}{F_1} = \dfrac{1}{2}.$ From equations (1) and (2)

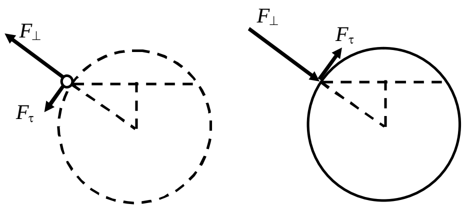
$\Rightarrow \theta < 30°. \left(\tan\theta = \dfrac{1}{2}, \tan 30^\circ = \dfrac{1}{\sqrt{3}} \text{ and } 2 > \sqrt{3}\right)$

Hence the total force on the ball is not normal to the ring. Further let us resolve the resultant force into components:

(1) Normal component $F_\perp = F \cos(30° - \theta)$, the component which is normal to the ring.

(2) Tangential component $F_\tau = F \sin(30° - \theta)$, the component which is tangential to the ring.

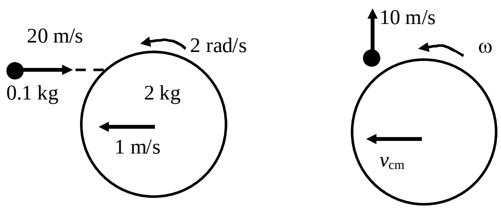
These two components and their reactions are shown in the figures below.



Force on ball by ring Force on ring by ball

From the above figure it can be inferred that during the collision there is a frictional force $F_\tau = F \sin(30° - \theta)$ on the ball by the ring. The reaction of this force acts on the ring. If the ball applies a frictional force on the ring during the collision, so does the ground. It can be shown that the point of contact of the ring has a tendency of motion relative to the ground.

Hence one may expect a frictional force on the ring by the ground. Moreover, during the collision, the normal force on the ring by the ground takes a large value. This can be inferred from the free body diagram of the ring.



Before collision After collision

From conservation of linear momentum,

$2 \times 1 \times 0.5 + 2 \times (0.5)^2 \times 2 - 1.0 \times 20 - 0.75$

$= 2 \times v_{cm} \times 0.5 + 2 \times (0.5)^2 \omega - 0.1 \times 10 \times 0.5 \times \dfrac{\sqrt{3}}{2}.$

From this single equation you cannot find out two unknown quantities v_{cm} and ω. Therefore, you cannot really conclude anything.

So the most important point is whether to consider the impulsive friction or not. The examiner (probably) did not consider it.